Math 301

HW 4 - Solutions.

① Let $x, y \in \mathbb{R}$ with $x<y$. Let $a_n = x + \frac{(y-x)}{2^n}$, then $(a_n)_{n\in\mathbb{N}}$ is a sequence in $(x,y]$ and $\lim a_n = x$ hence x is a limit point of $(x,y]$ but $x \notin (x,y]$ hence $(x,y]$ is not closed. Similarly $[x,y)$ is not closed. Are they open?: Given $\varepsilon>0$, then $(y-\varepsilon, y+\varepsilon) \subsetneq (x,y]$ hence y is not an interior point. Thus $(x,y]$ is not open. Similarly $[x,y)$ is not open.

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⑥ Suppose $A \subseteq \mathbb{R}$ both open and closed s.t. $A \neq \mathbb{R}$ and $A \neq \emptyset$. Then $A \cup A^c = \mathbb{R}$ and both A and A^c are open contradiction since $\mathbb{R}$ is connected.

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⑭ $A = \{0\} \cup (1,2] \cup \{4\}$, $\bar{A} = \{0\} \cup [1,2] \cup \{4\}$

$\mathring{A} = Int(A) = (1,2)$. Derived set: $[1,2]$. isolated points: $\{0, 4\}$.

A is not open 0 in not an interior point, A is not closed since 1 is a limit point but $1 \notin A$. A is not perfect since 0 and 4 are not limit points. A is not discrete since for any neighborhood N of $3/2$ $N \cap A$ contains points other that $3/2$.

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⑮ A' is closed. Let p be a limit point of A'. Given $\varepsilon>0$, then $\exists q \in A' \cap (p-\varepsilon, p+\varepsilon)$ s.t. $p \neq q$. Let $\alpha = \min\{|p-q|, \varepsilon - |p-q|\}$. Then, as $q \in A'$, $\exists r \in A \cap (q-\alpha, q+\alpha)$ s.t. $r \neq q$. Thus $r \in A \cap (p-\varepsilon, p+\varepsilon)$ so p is a limit point of A hence $p \in A'$. Therefore A' is closed.

(23) Let $(a_n)_{n\in\mathbb{N}}$ be a strictly increasing sequence, and $A = \{a_n : n \in \mathbb{N}\}$

Let $a_k \in A$. Since (a_n) is strictly increasing $a_{k-1} < a_k < a_{k+1}$.

Let $\varepsilon < \min\{a_k - a_{k-1}, a_{k+1} - a_k\}$, then $A \cap (a_k - \varepsilon, a_k + \varepsilon) = \{a_k\}$

Thus A is discrete.

A is closed iff (a_n) is unbounded.

($\Rightarrow$): Suppose A is closed. Assume that (a_n) is bounded then by monotone convergence thm. (a_n) converges, say to a.
Since A is closed, $a \in A$. Then $a = a_m$ for some $m \in \mathbb{N}$.
Then $a < a_n$ $\forall n > m$ but $a = \sup a_n$ (as (a_n) is increasing)
This contradiction shows that (a_n) is unbounded.

($\Leftarrow$): Suppose (a_n) is unbounded. Since $a_n < a_{n+1}$ $\forall n \in \mathbb{N}$
Let $O_n = (a_n, a_{n+1})$, and $F = (-\infty, a_0)$. Then $F \cup \bigcup_{n=0}^{\infty} O_n = A^c$ is open hence A is closed.

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(24) a) Since $\mathring{A}$ is the union of all open sets contained in A, $\mathring{A}$ is open.

b) $\mathring{A} = \bigcup_{\substack{B \subseteq A \\ B \text{ open}}} B$ so $\mathring{A} \subseteq A$.

c) Let $B \subseteq A$ open. Let $x \in B$. Since B is open $\exists \varepsilon > 0$ s.t. $(x-\varepsilon, x+\varepsilon) \subseteq B \subseteq A$ hence $x \in \mathring{A}$. Thus $B \subseteq \mathring{A}$.

d) Suppose $A = \mathring{A}$, then A is open since $\mathring{A}$ is open. Conversely suppose A is open then $A \subseteq \mathring{A}$ and by (b) $\mathring{A} \subseteq A$ so $A = \mathring{A}$.

(27) Let (X, d) be a metric space and Y be a nonempty subset of X.

Since d is a metric on X, $d(p,q)>0$ $\forall p,q \in X$ s.t. $p \neq q$ and $d(p,p)=0$ $\forall p \in X$. Hence $d(p,q)>0$ $\forall p,q \in Y$ s.t. $p \neq q$ and $d(p,p)=0$ $\forall p \in Y$.

$d(p,q)=d(q,p)$ $\forall p,q \in Y \subseteq X$ since d is a metric on X.

$d(p,q) \leq d(p,r)+d(r,q)$ for any $p,q,r \in Y \subseteq X$.

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(30) Let $f: X \longrightarrow \mathbb{R}$ be a one-to-one function on a nonempty set X.

Let $d(x,y)=|f(x)-f(y)|$ for every $x,y \in X$.

(i) $|f(x)-f(y)|=0$ iff $f(x)=f(y)$ iff $x=y$ as f is one-to-one.

Thus $d(x,y)=|f(x)-f(y)|>0$ $\forall x,y \in X$ with $x \neq y$.

(ii) $d(x,y)=|f(x)-f(y)|=|f(y)-f(x)|=d(y,x)$. $\forall x,y \in X$.

(iii) $d(x,y)=|f(x)-f(y)| \leq |f(x)-f(z)|+|f(z)-f(y)|=d(x,z)+d(z,y)$

$\forall x,y,z \in X$. Therefore d is a metric on X.

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(34) Let $m \in \mathbb{Z}^+$.

1) $d_1 : \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}$

$$d_1(x,y)=\sum_{i=1}^{m} |x_i - y_i|$$

(i) $d_1(x,y)=\sum_{i=1}^{m}|x_i-y_i|=0$ iff $|x_i-y_i|=0$ $\forall i$

iff $x_i=y_i$ $\forall i$ iff $x=y$.

$d_1(x,y)=\sum_{i=1}^{m}|x_i-y_i|>0$ if $x \neq y$. Indeed

since $x \neq y$, $x_i \neq y_i$ for some i hence $|x_i-y_i|>0$, so $d_1(x,y)>0$.

(ii) $d_1(x,y)=\sum_{i=1}^{m}|x_i-y_i|=\sum_{i=1}^{m}|y_i-x_i|=d_1(y,x)$. $\forall x,y \in \mathbb{R}^m$

(iii) $d_1(x,y)=\sum_{i=1}^{m}|x_i-y_i|=\sum_{i=1}^{m}|x_i-z_i+z_i-y_i| \leq \sum_{i=1}^{m}(|x_i-z_i|+|z_i-y_i|)$

$=\sum_{i=1}^{m}|x_i-z_i|+\sum_{i=1}^{m}|z_i-y_i|=d_1(x,z)+d_1(z,x)$.

Thus d_1 is a metric on $\mathbb{R}^m$

1) $d_2 : \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathbb{R}$

$d_2(x,y) = \sqrt{\sum_{i=1}^{m} |x_i - y_i|^2}$

i) $d_2(x,y) = \sqrt{\sum_{i=1}^{m} |x_i - y_i|^2} = 0$ iff $\sum_{i=1}^{m} |x_i - y_i|^2 = 0$ iff $|x_i - y_i| = 0 \ \forall i$ iff $x_i = y_i \ \forall i$ iff $x = y$.

$d_2(x,y) = \sqrt{\sum_{i=1}^{m} |x_i - y_i|^2} > 0$ if $x \neq y$ since $|x_i - y_i|^2 > 0$ for some i.

(ii) $d_2(x,y) = \sqrt{\sum_{i=1}^{m} |x_i - y_i|^2} = \sqrt{\sum_{i=1}^{m} |y_i - x_i|^2} = d_2(y,x)$.

(iii) Let $a_i = x_i - z_i$, $b_i = z_i - y_i$. By Cauchy-Schwarz inequality

$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \leq \sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2$ so $\sum_{k=1}^{n} (a_k + b_k)^2 = \sum_{k=1}^{n} a_k^2 + 2\sum_{k=1}^{n} a_k b_k + \sum_{k=1}^{n} b_k^2$

$\leq \sum_{k=1}^{n} a_k^2 + 2\sqrt{\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2} + \sum_{k=1}^{n} b_k^2 = \left(\sqrt{\sum_{k=1}^{n} a_k^2} + \sqrt{\sum_{k=1}^{n} b_k^2}\right)^2$

This is equivalent to $d(x,y) \leq d(x,z) + d(z,y)$.

Therefore d_2 is a metric on $\mathbb{R}^m$.

3) $d_\infty(x,y) = \max\{|x_i - y_i| : 1 \leq i \leq m\}$

(i) $d_\infty(x,y) = 0$ iff $|x_i - y_i| = 0 \ \forall i$ iff $x_i = y_i \ \forall i$ iff $x = y$

$d_\infty(x,y) > 0$ if $x \neq y$ since $|x_i - y_i| > 0$ for some i.

(ii) $d_\infty(x,y) = d_\infty(y,x)$ since $|x_i - y_i| = |y_i - x_i|$.

(iii) $d_\infty(x,y) = \max\{|x_i - y_i| : 1 \leq i \leq m\} = \max\{|x_i - z_i + z_i - y_i| : 1 \leq i \leq m\}$

$\leq \max\{|x_i - z_i| + |z_i - y_i| : 1 \leq i \leq m\} \leq \max\{|x_i - z_i| : 1 \leq i \leq m\}$

$+ \max\{|z_i - y_i| : 1 \leq i \leq m\} = d_\infty(x,z) + d_\infty(z,y)$.

Thus d_∞ is a metric on $\mathbb{R}^m$.

(35) Let S be any nonempty set. Let $B(S) = \{f : S \to \mathbb{R} : f \text{ is bounded}\}$

Let $\rho : B(S) \times B(S) \longrightarrow \mathbb{R}$

$$\rho(f,g) = \sup\{|f(s) - g(s)| : s \in S\}$$

Since f and g are bounded sup exists.

i) $\rho(f,g) = \sup\{|f(s)-g(s)| : s \in S\} = 0$ iff $|f(s)-g(s)| = 0 \quad \forall s \in S$

(Since $|f(s)-g(s)| \geq 0 \quad \forall s \in S$) iff $f(s) = g(s) \quad \forall s \in S$

iff $f = g$. $\rho(f,g) > 0$ if $f \neq g$ since $\exists s \in S : f(s) \neq g(s)$

so $|f(s) - g(s)| > 0$ hence $\sup\{|f(s)-g(s)| : s \in S\} > 0$.

(ii) $\rho(f,g) = \rho(g,f)$ since $|f(s) - g(s)| = |g(s) - f(s)|$

(iii) $\rho(f,g) = \sup\{|f(s) - g(s)| : s \in S\} = \sup\{|f(s) - h(s) + h(s) - g(s)| : s \in S\}$

$\leq \sup\{|f(s) - h(s)| + |h(s) - g(s)| : s \in S\}$

$\leq \sup\{|f(s) - h(s)| : s \in S\} + \sup\{|h(s) - g(s)| : s \in S\}$

$= \rho(f,h) + \rho(h,g)$.

Thus ρ is a metric on $B(S)$.