Math 301

HW 5 solutions

(5) Take discrete metric, and consider $B_1(x) = \{y \in X : d(x,y) < 1\} = \{x\}$

Then $\bar{B}_1(x) = \{y \in X : d(x,y) \leq 1\} = X$ and $\overline{B_1(x)} = \{x\}$ as if $y \in X \setminus \{x\}$ $B_{1/2}(y) \cap B_1(x) = \emptyset$. Hence $\bar{B}_1(x) \neq \overline{B_1(x)}$.

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(6) Let A be open and $A \cap B = \emptyset$. Suppose $A \cap \bar{B} \neq \emptyset$ and let $x \in A \cap \bar{B}$

Then $x \in A$ and as A is open $\exists \varepsilon > 0 : B_\varepsilon(x) \subseteq A$. Since $x \in \bar{B}$ $\exists y \in B \cap B_\varepsilon(x)$ so $y \in A$ hence $y \in A \cap B$ contrary to $A \cap B = \emptyset$.

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(13) $x \in \bar{A} \iff \inf\{d(x,a) : a \in A\} = 0$.

($\Rightarrow$): Let $x \in \bar{A}$. Let $\alpha = \inf\{d(x,a) : a \in A\}$. Given $\varepsilon > 0$. As $x \in \bar{A}$, $\exists y \in A$ s.t. $d(x,y) < \varepsilon$. Hence $\alpha < \varepsilon$. As $d(x,a) \geq 0$ $\forall a \in A$, $\alpha \geq 0$ so $0 \leq \alpha < \varepsilon$. As ε is arbitrary $\alpha = 0$.

($\Leftarrow$): Suppose $\inf\{d(x,a) : a \in A\} = 0$. Given $\varepsilon > 0$ $\exists a \in A$, $d(x,a) < \varepsilon$, as $\inf\{d(x,a) : a \in A\} = 0$. Hence $B_\varepsilon(x) \cap A \neq \emptyset$. So $x \in \bar{A}$.

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(15) $\mathring{\mathbb{Q}} = \emptyset$ as every interval in $\mathbb{R}$ contains a point from $\mathbb{R} \setminus \mathbb{Q}$.

$\bar{\mathbb{Q}} = \mathbb{R}$ as $\mathbb{Q}$ is dense in $\mathbb{R}$. $\operatorname{ext} \mathbb{Q} = \operatorname{int}(\mathbb{R} \setminus \mathbb{Q}) = \emptyset$ as every interval in $\mathbb{R}$ contains a point from $\mathbb{Q}$. $\partial \mathbb{Q} = \bar{\mathbb{Q}} \setminus \mathring{\mathbb{Q}} = \mathbb{R}$.

$\mathring{\mathbb{N}} = \emptyset$, $\bar{\mathbb{N}} = \mathbb{N}$, $\operatorname{ext} \mathbb{N} = \operatorname{int}(\mathbb{R} \setminus \mathbb{N}) = (-\infty, 0) \cup \bigcup_{n=0}^{\infty} (n, n+1)$.

$\partial \mathbb{N} = \bar{\mathbb{N}} \setminus \mathring{\mathbb{N}} = \mathbb{N}$.

$I = [a,b]$, then $\mathring{I} = (a,b)$, $\bar{I} = [a,b]$, $\operatorname{ext} I = (-\infty, a) \cup (b, \infty)$, $\partial I = \{a,b\}$

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Similarly, for any kind of interval we have $\mathring{I} = (a,b)$, $\bar{I} = [a,b]$ $\operatorname{ext} I = (-\infty, a) \cup (b, \infty)$, $\partial I = \{a,b\}$.

(20) $\partial A = \bar{A} \setminus \mathring{A}$. Hence $\partial A \subseteq \bar{A}$.

$$A \cup \partial A = A \cup (\bar{A} \setminus \mathring{A}) = A \cup (\bar{A} \cap (\mathring{A})^c) = \underbrace{(A \cup \bar{A})}_{\bar{A}} \cap \underbrace{(A \cup (\mathring{A})^c)}_{X}$$

$$= \bar{A} \cap X = \bar{A}$$

(22) $\partial A = \bar{A} \setminus \mathring{A} = \bar{A} \cap (\mathring{A})^c$. As $\mathring{A}$ is open, $(\mathring{A})^c$ is closed. Since $\bar{A}$ is also closed, $\partial A = \bar{A} \cap (\mathring{A})^c$ is closed.