Math 301
HW9 - Solutions

(2) (a) $[0,1]$ is a completion of $(0,1)$ with the absolute value metric as $[0,1]$ is complete and $\overline{(0,1)} = [0,1]$.

(b) Since every discrete metric space is complete, $(0,1)$ is already complete.

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(3) As (X,d) is complete and totally bounded, it is compact. Let $f:(X,d) \longrightarrow (Y,\rho)$ be a continuous function. Let $\varepsilon > 0$. and $x \in X$. Since f is continuous, $\exists\, \delta_x > 0$ s.t. $\rho(f(x), f(y)) < \varepsilon/2$ whenever $d(x,y) < \delta_x$. Now for each $x \in X$, there is δ_x corresponding to x. Since $X = \bigcup_{x\in X} B_{\delta_x/2}(x)$, the family $\{B_{\delta_x/2}(x)\}_{x\in X}$ forms an open cover of X. Since X is compact, there is a finite subcover, say $X \subseteq \bigcup_{i=1}^{n} B_{\delta_{x_i}/2}(x_i)$. Put $\delta = \frac{1}{2}\min\{\delta_{x_1}, \dots, \delta_{x_n}\}$. Let $x, y \in X$ be s.t. $d(x,y) < \delta$. Then $x \in B_{\delta_{x_i}/2}(x_i)$ for some $1 \le i \le n$, i.e. $d(x, x_i) < \delta_{x_i}/2$. We have $d(y, x_i) \le d(x,y) + d(x, x_i) < \delta + \delta_{x_i}/2 < \delta_{x_i}$. Then $\rho(f(x), f(y)) \le \rho(f(x), f(x_i)) + \rho(f(x_i), f(y)) < \varepsilon/2 + \varepsilon/2 = \varepsilon$. Thus f is uniformly continuous.

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(5) Suppose that A is compact. Let $x \in X$. Consider the map

$$f_x : A \longrightarrow \mathbb{R}, \quad a \longmapsto d(x,a)$$

$d(x,a) \le d(x,b) + d(a,b)$ and $d(x,b) \le d(x,a) + d(a,b)$. Hence

(*) $|d(x,a) - d(x,b)| \le d(a,b)$. Given $\varepsilon > 0$, choose $\delta = \varepsilon$. then by (*) $|f_x(a) - f_x(b)| < \varepsilon$ whenever $d(a,b) < \delta$. Hence f_x is continuous. Since A is compact f_x attains its infimum, so $\exists\, y \in A : f_x(y) = d(x,y) = \inf_{a\in A} d(x,a) = d(x,A)$.

⑥ (a) Consider $A = \{(x, 1/x) : x \in \mathbb{R}\} \subseteq \mathbb{R}^2$ and $B = \{(x, 0) : x \in \mathbb{R}\} \subseteq \mathbb{R}^2$

Then $\inf_{a \in A} d(a, B) = 0$ as $\frac{1}{x} \longrightarrow 0$.

(b) Let A and B are disjoint, A is closed and B is compact. Suppose that the distance between A and B is zero. Consider the function $f : B \longrightarrow \mathbb{R}$, $b \longmapsto d(b, A)$. We show that f is continuous on B.

Let $b_0 \in B$ and $\varepsilon > 0$. Let $b_1 \in B$. $d(b_0, a) \leq d(b_0, b_1) + d(b_1, a)$ and $d(b_1, a) \leq d(b_1, b_0) + d(b_0, a)$ hence $\inf_{a \in A} d(b_0, a) \leq d(b_0, b_1) + \inf_{a \in A} d(b_1, a)$ and $\inf_{a \in A} d(b_1, a) \leq d(b_1, b_0) + \inf_{a \in A} d(b_0, a)$. Then $\left| \inf_{a \in A} d(b_0, a) - \inf_{a \in A} d(b_1, a) \right| < d(b_1, b_0)$. i.e. $|f(b_0) - f(b_1)| < d(b_1, b_0)$. Choose $\delta = \varepsilon$ so $|f(b_0) - f(b)| < \varepsilon$ whenever $d(b_0, b) < \delta$. Therefore f is continuous and as B is compact f attains its infimum. Hence there exists $b' \in B$ s.t. $d(b', A) = \inf_{b \in B} d(b, A) = 0$. i.e. $\inf_{a \in A} d(b', a) = 0$

Given $\varepsilon > 0$ $\exists a \in A : d(b', a) < \varepsilon$ hence b' is a limit point of A and as A is closed, $b' \in A$ but then $b' \in A \cap B$, contradiction. Thus the distance between A and B can not be zero.

(8) Suppose X is compact and $\phi : X \longrightarrow X$ is a mapping satisfies.

$$(*) \quad d(\phi(x), \phi(y)) < d(x,y) \text{ whenever } x \neq y.$$

Consider the function $\theta : X \longrightarrow \mathbb{R}$

$$x \longmapsto d(x, \phi(x))$$

θ is continuous on X. Indeed, let $x \in X$, $\varepsilon > 0$. Then

$d(x, \phi(x)) \leq d(x,y) + d(y, \phi(y)) + d(\phi(y), \phi(x))$ by triangle inequality.

Then $d(x,\phi(x)) - d(y, \phi(y)) \leq d(x,y) + d(\phi(y), \phi(x)) < 2\, d(x,y)$

Similarly $d(y, \phi(y)) - d(x, \phi(x)) \geq -d(x,y) - d(\phi(y), \phi(x)) > -2d(x,y)$

Then $|d(y,\phi(y)) - d(x, \phi(x))| < 2\, d(x,y)$ i.e. $|\theta(x) - \theta(y)| < 2d(x,y)$

Choose $\delta = \varepsilon/2$ then $|\theta(x) - \theta(y)| < \varepsilon$ whenever $d(x,y) < \delta$.

Hence θ is continuous. Since X is compact, θ attains its infimum. i.e. $\exists x_0 \in X : \theta(x_0) = d(x_0, \phi(x_0)) = \inf_{x \in X} d(x, \phi(x))$.

Then $d(x_0, \phi(x_0)) \leq d(x, \phi(x))\ \forall x \in X$ and in particular

$d(x_0, \phi(x_0)) \leq d(\phi(x_0), \phi(\phi(x_0)))$ hence by $(*)$ $x_0 = \phi(x_0)$

and so x_0 is a fixed point of ϕ. If x_1 is another fixed point and $x_1 \neq x_0$ then by $(*)$ $d(x_0, x_1) > d(\phi(x_0), \phi(x_1)) = d(x_0, x_1)$

contradiction. Thus ϕ has a unique fixed point.