Math 301 - Problem Set # 7 - Fall 2010

Homework Problems: 1, 2, 8, 12, 15.

In the following problems (X, d), (Y, ρ) and (Z, u) are assumed to be metric spaces, $A, B \subseteq X$, f, g are mappings from X to Y, and h is a mapping from Y to Z.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & , \quad x \in \mathbb{Q} \\ 0 & , \quad x \notin \mathbb{Q} \end{cases}$$

   Is there a point at which f is continuous?

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Prove that there exists a constant c such that $f(x) = cx$ for every $x \in \mathbb{R}$.

3. Let $f : X \to \mathbb{R}$ be defined by $f(x) = \inf\{d(x, a) : a \in A\}$ for every $x \in X$. Prove that f is continuous. Also prove that f is uniformly continuous if A consists of a single point.

4. Prove that a mapping between metric spaces is continuous iff the preimage of every closed set is closed.

5. Prove that f is continuous iff $f(\bar{A}) \subseteq \overline{f(A)}$ for every subset A of X.

6. Prove that f is continuous iff $f^{-1}(E^o) \subseteq (f^{-1}(E))^o$ for every subset E of Y.

7. Suppose that f is a bijection. Prove that the following are equivalent.

   (a) f is an open mapping.

   (b) f is a closed mapping.

   (c) f^{-1} is continuous.

8. In each of the following, give an example of a mapping between metric spaces with the given property or explain why no such example exists.

   (a) A continuous mapping which is not open.

   (b) A continuous mapping which is not closed.

   (c) An open mapping which is not continuous.

   (d) A closed mapping which is not continuous.

   (e) A closed mapping which is not open.

   (f) An open mapping which is not closed.

(g) A continuous open mapping which is not a homeomorphism.

(h) A continuous closed mapping which is not a homeomorphism.

(i) A homeomorphism which is not open.

(j) A homeomorphism which is not closed.

(k) A continuous bijection which is not open.

(l) A continuous bijection which is not closed.

(m) A continuous bijection which is not a homeomorphism.

9. Prove that if f and h are homeomorphisms, then so is $h \circ f$, and, as a consequence, if X is homeomorphic to Y and Y is homeomorphic to Z, then X is homeomorphic to Z.

10. Give an example of a subspace of $(\mathbb{R}, d_1)$, where d_1 is the absolute value metric, which is homeomorphic to $(\mathbb{R}, d_1)$.

11. Prove that any two open intervals in $\mathbb{R}$ are homeomorphic when considered with the absolute value metric.

12. Give an example of a pair of homeomorphic metric spaces such that one is complete and the other is not.

13. Suppose that f is an isometry. Prove the following.

    (a) f is one-to-one.

    (b) f is uniformly continuous.

    (c) $f^{-1} : f(X) \to X$ is an isometry.

    (d) If f is onto, then it is a homeomorphism.

14. Suppose that f and g are continuous and A is dense in X. Prove that if $f(a) = g(a)$ for every $a \in A$, then $f = g$, i.e. $f(x) = g(x)$ for every $x \in X$.

15. Consider the function $f_A : X \to \mathbb{R}$ defined by $f_A(x) = \inf\{d(x, a) : a \in A\}$ for every $x \in X$. f_B is defined similarly. *the distance between* A *and* B is defined to be $\inf\{f_A(x) : x \in B\} = \inf\{f_B(x) : x \in A\}$. Prove the following.

    (a) f_A is uniformly continuous regardless of A.

    (b) $\bar{A} = f_A^{-1}(0)$.

    (c) If $A \cap B \neq \emptyset$, then the distance between A and B is 0.

    (d) There may be disjoint subsets of X with 0 distance between them.

    (e) If A and B are disjoint and closed in X, then the function $g : X \to \mathbb{R}$ defined by $g(x) = f_A(x) - f_B(x)$ for every $x \in X$ is uniformly continuous and, as a consequence, $g^{-1}(0, \infty)$ and $g^{-1}(-\infty, 0)$ are disjoint open subsets of X containing A and B, respectively.

16. Using the function f_A defined in the previous problem, prove that a closed subset A of X is equal to the intersection of a sequence of open sets in X.

17. Prove that any open subset of X is equal to the union of a sequence of closed subsets.

18. Let $p \in [0, 1)$. Considering the function $P : [0, \infty) \to \mathbb{R}$ defined by $P(x) = x^p$, prove that $\lim((n+1)^p - n^p) = 0$.

19. Consider $X = [1, \infty)$, $A = (1, \infty)$ and $Y = (0, 1)$ with the absolute value metric and $f : A \to Y$ be defined by $f(a) = 1/a$ for every $a \in A$. Check that A is dense in X and f is uniformly continuous. Can you find a uniformly continuous extension $\bar{f} : X \to Y$ of f? How does your answer relate to the "Uniform Extension Theorem" we proved in class ? Can you find a convergent sequence (a_n) in A such that $(f(a_n))$ is not convergent in Y?