Math 301 - Problem Set # 8 - Fall 2010

Homework Problems: 2, 5, 6, 8, 9, 11.

In the following problems (X, d), (Y, ρ) are assumed to be metric spaces, f is a mapping from X to Y, and (x_n) is a sequence in X.

1. Prove that if f is uniformly continuous and (x_n) is Cauchy, then $(f(x_n))$ is also Cauchy.
2. Give an example of a continuous function on a subset of $\mathbb{R}$ which sends a Cauchy sequence to a "non-Cauchy" sequence.
3. Let A be closed in X and $\varphi : A \to \mathbb{R}$ be a continuous function. Prove that φ can be extended to a continuous function from X to $\mathbb{R}$. *Hint: Use Tietze Extension Theorem, Problem 8 in HW9 and Urysohn's Lemma.*
4. Prove that any Lipschitz mapping is uniformly continuous.
5. Prove that the square-root function is uniformly continuous on $(0, \infty)$, but Lipschitz on (a, ∞) only when $a > 0$.
6. Prove that any real-valued differentiable function φ on $\mathbb{R}$ with bounded derivative is a Lipschitz mapping. Moreover, if the supremum of $|\varphi'(x)|$ is less than 1, then φ is a contraction mapping. *Hint: Use the Mean Value Theorem.*
7. Suppose that (X, d) is complete and $\varphi : X \to X$ is a mapping. Prove that if φ^n is a contraction mapping for a positive integer n, then φ has a unique fixed point.
8. Using Banach Fixed Point Theorem, prove that the following initial-value problem has a unique solution y on the interval $[0, 1]$

$$y' = F(x, y) \text{ and } y(0) = 0,$$

   if $F : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function whose partial derivative F_2 with respect to the second variable is between $-1/2$ and $1/2$.
9. Give an example of a mapping $\varphi : X \to X$ such that $d(\varphi(x), \varphi(x')) < d(x, x')$ for every $x \neq x' \in X$ which has no fixed points.
10. Let $\mathcal{X}$ be the cartesian product of $X_1, \ldots, X_k$, where $(X_1, d_1), \ldots, (X_k, d_k)$ are metric spaces. Let ρ_∞ and ρ_p be functions on $\mathcal{X} \times \mathcal{X}$ for $p \geq 1$ as defined in class.
    (a) Prove that ρ_∞ is a metric on $\mathcal{X}$.
    (b) Prove that ρ_1 is a metric on $\mathcal{X}$.

(c) Prove that for each $p > 1$ ρ_p satisfies the first two conditions in the definition of a metric.

(d) Prove the following inequalities.

$$\rho_1(x, y) \leq k\rho_\infty(x, y)$$

$$\rho_p(x, y) \leq \rho_1(x, y)$$

$$\rho_\infty(x, y) \leq \rho_p(x, y)$$

for every $x, y \in \mathcal{X}$, where $p \geq 1$.

(e) Prove that the metric ρ_∞ is equivalent to ρ_p for every $p \geq 1$.

(f) Prove that, for each $i \in \{1, \ldots, k\}$, the projection $\pi_i : \mathcal{X} \to X_i$ defined by $\pi_i(x) = x_i$ for every $x = (x_1, \ldots, x_k) \in \mathcal{X}$ is uniformly continuous.

(g) Prove that a mapping $\varphi : Y \to \mathcal{X}$ is continuous iff $\pi_i \circ \varphi$ is continuous for every i.

(h) Let $\mathcal{A}$ be a subset of $\mathcal{X}$. Prove that $\mathcal{A}$ is bounded iff $\pi_i(\mathcal{A})$ is bounded in X_i for every i.

11. A metric space is called *separable* if it has a countable dense subset. Prove that the cartesian product of finitely many separable metric spaces is also separable.

12. Considering the absolute value metric on $\mathbb{R}$ and any of the metrics ρ_p, prove that $\mathbb{R}^k$ is complete.