Math 301

HW-10, Solutions

(1) Suppose $\bigcap_{n=0}^{\infty} A_n = \emptyset$, then put $B_n = A_n^c$, hence $\bigcup_{n=0}^{\infty} B_n = \bigcup_{n=0}^{\infty} A_n^c = \left(\bigcap_{n=0}^{\infty} A_n\right)^c = X$ as $\bigcap_{n=0}^{\infty} A_n = \emptyset$. Hence (B_n) is an open cover of X and as X is compact there exists $B_{n_1}, \dots, B_{n_k}$ s.t. $X = \bigcup_{i=1}^{k} B_{n_i} = \bigcup_{i=1}^{k} A_{n_i}^c = \left(\bigcap_{i=1}^{k} A_{n_i}\right)^c$ so $\bigcap_{i=1}^{k} A_{n_i} = \emptyset$ but if $\max_i (n_i) = k$ as $A_{n+1} \subseteq A_n$ $\bigcap_{i=1}^{k} A_{n_i} = A_k \neq \emptyset$ by hypothesis. Thus $\bigcap_{n=0}^{\infty} A_n \neq \emptyset$.

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(3) Let $f_n : [0,1] \longrightarrow \mathbb{R}$ be defined by $f_n(x) = x^n$. Let $x \in [0,1[$, then $f_n(x) = x^n \longrightarrow 0$ as $n \longrightarrow \infty$ and if $x = 1$, then $f_n(x) \longrightarrow 1$. Hence $(f_n(x))$ is convergent on $[0,1]$. Put $f(x) = \begin{cases} 0 & x \in [0,1[\\ 1 & x = 1 \end{cases}$

then $f_n(x) \longrightarrow f(x)$ pointwise on $[0,1]$, $f_n(x)$ is continuous for each n and $f(x)$ is not continuous, hence convergence is not uniform.

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(10) Suppose that A and B are connected.

a) Suppose $\bar{A} \cap B \neq \emptyset$, and suppose $A \cup B \subseteq O_1 \cup O_2$, where O_1, O_2 are nonempty, open, disjoint subsets of X. Then $B \subseteq O_1 \cup O_2$ and as B is connected $B \subseteq O_1$ or $B \subseteq O_2$. We may assume $B \subseteq O_1$ ($B \subseteq O_2$ is similar). Let $a \in \bar{A} \cap B$, then $a \in O_1$ and as O_1 is open $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq O_1$. Since $a \in \bar{A}$, $B_\varepsilon(x) \cap (A \setminus \{a\}) \neq \emptyset$ i.e. $A \cap O_1 \neq \emptyset$. Since A is connected and $A \subseteq O_1 \cup O_2$, $A \subseteq O_1$ or $A \subseteq O_2$. As $A \cap O_1 \neq \emptyset$ $A \subseteq O_1$. Thus we have $A \cup B \subseteq O_1$ and so $A \cup B$ is connected.

b) Let $A=(0,1)$ and $B=(1,2)$. Then $\bar{A}\cap\bar{B}=\{1\}$ but $A\cup B=(0,1)\cup(1,2)$ is not connected.

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⑰ Suppose that X is connected and not bounded, and $x_0\in X$. Let $r>0$. For a contradiction assume there does not exists $x\in X$ with $d(x,x_0)=r$. Put $O_1=\{x\in X: d(x,x_0)<r\}$ and $O_2=\{x\in X; d(x,x_0)>r\}$ Then $X=O_1\cup O_2$ and $X\cap O_1\neq\emptyset$ as $x_0\in X\cap O_1$ and $X\cap O_2\neq\emptyset$ as X is unbounded. Thus X is disconnected contrary to hypothesis. Therefore $\exists x\in X: d(x,x_0)=r$.

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⑲ (a) $A=\{\sin(1/n): n\in\mathbb{Z}_+\}\subseteq\mathbb{R}$. A is not compact as 0 is a limit point $(\sin(1/n)\longrightarrow 0$ as $n\to\infty)$ but $0\notin A$. Since A is countable A is not connected as A does not contain an interval.

(b) $B=\{(x,e^x): 0<x<1\}\subseteq\mathbb{R}^2$. Let $f:\mathbb{R}\longrightarrow\mathbb{R}$ be a continuous function. Consider the function $g:\mathbb{R}\longrightarrow\mathbb{R}^2$, $g(x)=(x,f(x))$. We show that g is continuous. Let $x\in\mathbb{R}$ and $(x_n)_{n\in\mathbb{N}}$ be a sequence converging to x. Then $\lim\limits_{n\to\infty} g(x_n)=\lim\limits_{n\to\infty}(x_n,f(x_n))=(x,f(x))=g(x)$ as f is continuous. Thus g is continuous. Now as e^x is continuous and $(0,1)$ is connected B is connected. Since $(1,e)$ is a limit point but $(1,e)\notin B$, B is not compact.

(c) $C=\{(x,y,z): x^2+y^2\leq 1\}\cap\{(x,y,z): z^2+y^2\leq 4\}\subseteq\mathbb{R}^3$. C being bounded and intersection of two closed sets, it is compact.
C is clearly connected.

(d) $D = \{z : z = x^2 \sin y,\ x^2+y^2 \leq 1\} \subseteq \mathbb{R}$. Consider $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $f(x,y) = x^2 \sin y$. It is easy to see that f is continuous. Let $S = \{(x,y) : x^2+y^2 \leq 1\}$. Then $D = f(S)$. As S is connected and compact, $f(S) = D$ is also connected and compact since f is continuous.

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(26) Let $E = \{(0,y) : -1 \leq y \leq 1\}$ and $F = \{(x, \sin(1/x)) : 0 < x \leq 1\}$. and $A = E \cup F$. We show that A is connected but not path connected. Since $f(x) = (x, \sin(1/x))$ is continuous on $(0,1]$ and $(0,1]$ is connected, F is connected. As $\overline{F} = E \cup F = A$, A is also connected. Let $a = (0,0)$ and $b = (1, \sin 1)$. Then $a, b \in A$. We show that there is no path joining a and b in A. i.e. there is no continuous function $\gamma : [0,1] \longrightarrow A$ s.t. $\gamma(0) = a$ and $\gamma(1) = b$. Assume that there exists such a γ. Then, for $t > 0$, we have $\gamma(t) \in F$ i.e. $\gamma(t) = (t, \sin(1/t))$. Since $\sin(1/t)$ is not continuous at zero, γ can not be continuous at zero, contradiction. Thus A is not path-connected.