HW 2
Solutions

6- Suppose $\lim a_n < \lim b_n$. Let $A = \lim a_n$ and $B = \lim \beta_n$.

Choose $\varepsilon = \frac{B-A}{2}$. Since $a_n \longrightarrow A$, $\exists N_1 \in \mathbb{N} : |A - a_n| < \varepsilon \ \forall n \geq N_1$

and since $b_n \longrightarrow B$, $\exists N_2 \in \mathbb{N} : |b_n - B| < \varepsilon \ \forall n \geq N_2$.

Then $-\varepsilon - A < -a_n < \varepsilon - A$ and $-\varepsilon + B < b_n < \varepsilon + B \ \forall n \geq \max\{N_1, N_2\}$

so $-2\varepsilon + B - A < b_n - a_n$. Put $\varepsilon = \frac{B-A}{2}$, we get $b_n - a_n > 0$

hence $b_n > a_n \ \forall n \geq \max\{N_1, N_2\}$

Counter example for the converse: Take $a_n = 1 - \frac{1}{n}$ and $b_n = 1 + \frac{1}{n}$

then $a_n < b_n \ \forall n \geq 1$ but $\lim a_n = \lim b_n = 1$

10- $\lim \left(\sqrt{n+1} - \sqrt{n}\right) = \lim \left(\frac{\left(\sqrt{n+1} - \sqrt{n}\right)\left(\sqrt{n+1} + \sqrt{n}\right)}{\sqrt{n+1} + \sqrt{n}} \right) = \lim \left(\frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} \right)$

$= \lim \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0$. $\left| \frac{1}{\sqrt{n+1} + \sqrt{n}} \right| \leq \left| \frac{1}{n} \right| < \varepsilon \ \forall n \geq N$ for $N > \frac{1}{\varepsilon}$.

12- Since (b_n) is bounded $\exists M \in \mathbb{R} : |b_n| \leq M \ \forall n \geq 1$.

Given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $|a_n| < \frac{\varepsilon}{M} \ \forall n \geq N$.

Then $|a_n b_n| = |a_n||b_n| \leq |a_n| M < \frac{\varepsilon}{M} \cdot M = \varepsilon \ \forall n \geq N$.

Thus $\lim a_n b_n = 0$.

Counterexample for (b_n) is not bounded: Take $a_n = \frac{1}{n}$ and $b_n = n^2$

then $\lim a_n = 0$ but $\lim a_n b_n = \lim n \neq 0$.

17- $(a_n)_{n\in\mathbb{N}}$ is given by $a_0 = 1$ $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$. Observe that $2a_n a_{n+1} = a_n^2 + 2$. hence a_n satisfies the equation $a_n^2 - 2a_n a_{n+1} + 2 = 0$ so this equation has a real root which implies that the discriminant $4a_{n+1}^2 - 8 \geq 0$, so $a_{n+1}^2 \geq 2$ $\forall n \geq 1$ and $a_n^2 \geq 2$ $\forall n \geq 2$. Therefore (a_n) is bounded.

Consider

$$a_n - a_{n+1} = a_n - \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) = \frac{a_n^2 - 2}{2a_n} \geq 0 \quad \forall n \geq 2$$

as $a_n^2 \geq 2$. Hence $a_n \geq a_{n+1}$, $\forall n \geq 2$ and so (a_n) is monotone decreasing. By Monotone Convergence Theorem $\lim a_n$ exists. Let $a = \lim a_n$. Then by the properties of limit $a = \lim a_{n+1} = \lim \frac{1}{2}\left(a_n + \frac{2}{a_n}\right) = \frac{1}{2}\left(a + \frac{2}{a}\right)$

hence $a = \frac{1}{2}\left(a + \frac{2}{a}\right)$ so $a^2 = 2$ and $a = \sqrt{2}$.

22- Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of positive real numbers.

Suppose that $\lim \frac{a_{n+1}}{a_n} = L$. Choose $\beta > L$. Then as

$\lim \frac{a_{n+1}}{a_n} = L$, $\exists N \in \mathbb{N}$ s.t. $\frac{a_{n+1}}{a_n} \leq \beta \quad \forall n \geq N$ Hence

$$a_{N+1} \leq \beta a_N$$

$$a_{N+2} \leq \beta a_{N+1} \leq \beta^2 a_N$$

$$\vdots$$

$$a_{N+k} \leq \beta^k a_N$$

So $\quad a_n \leq a_N \beta^{n-N} \quad \forall n \geq N$. Then $\sqrt[n]{a_n} \leq \sqrt[n]{a_N \beta^{-N}} \cdot \beta$, taking

the limit, $\lim \sqrt[n]{a_n} \leq \lim \sqrt[n]{a_N \beta^{-N}} \cdot \beta = \beta$ as $a_N \beta^{-N} > 0$ and

for any $c > 0$ $\lim \sqrt[n]{c} = 1$. Hence $\lim \sqrt[n]{a_n} \leq \beta$. Since the

last inequality holds for any $\beta > L$, it also holds for L

so $\lim \sqrt[n]{a_n} \leq L$.

For the inverse inequality choose $\beta < L$, then $\exists N \in \mathbb{N}$ s.t

$\frac{a_{n+1}}{a_n} \geq \beta \quad \forall n \geq N$ so similarly as above $a_n \geq \beta^{n-N} a_N \quad \forall n \geq N$.

Then $\sqrt[n]{a_n} \geq \sqrt[n]{\beta^{-N} a_N} \cdot \beta \quad \forall n \geq N$. Taking the limit, $\lim \sqrt[n]{a_n} \geq \beta$.

hence $\lim \sqrt[n]{a_n} \geq L$. Together with the above result we have

$\lim \sqrt[n]{a_n} = L$. Consider $\lim \sqrt[n]{a_n} = \lim \sqrt[n+1]{a_{n+1}} = \lim \left(\sqrt[n+1]{\frac{a_{n+1}}{a_n}} \sqrt[n+1]{a_n} \right)$

$= \lim \sqrt[n+1]{\frac{a_{n+1}}{a_n}} \cdot \lim \sqrt[n+1]{a_n}$. Since $\frac{a_{n+1}}{a_n}$ is bounded, $\lim \sqrt[n+1]{\frac{a_{n+1}}{a_n}} = 1$

hence $\lim \sqrt[n]{a_n} = \lim \sqrt[n+1]{a_n} = L$.

Counter example for the converse:

Take $a_n = \begin{cases} \frac{1}{2^{n+1}} & \text{if } n \text{ is even} \\ \frac{1}{2^{n-1}} & \text{if } n \text{ is odd} \end{cases}$

$\lim\limits_{\substack{n \to \infty \\ n \text{ is even}}} \sqrt[n+1]{a_n} = \frac{1}{2}$ and $\lim\limits_{\substack{n \to \infty \\ n \text{ is odd}}} \sqrt[n+1]{a_n} = \frac{1}{2}$, hence $\lim \sqrt[n+1]{a_n} = \frac{1}{2}$.

$\frac{a_{n+1}}{a_n} = \begin{cases} 2 & \text{if } n \text{ is even.} \\ \frac{1}{8} & \text{if } n \text{ is odd} \end{cases}$ hence $\lim \frac{a_{n+1}}{a_n}$ does not exist.