Math 301

HW-8 Solutions.

② $f:(0,1) \longrightarrow \mathbb{R}$, $f(x) = \frac{1}{x}$. Let $a_n = \frac{1}{n}$ $n \geq 2$. Then $(a_n)_{n\in\mathbb{N}}$ is a cauch

sequence but $f(a_n) = n$ is not a cauchy sequence as it diverges.

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⑤ $f:(0,\infty) \longrightarrow \mathbb{R}$, $f(x) = \sqrt{x}$. Given $\varepsilon > 0$, choose $\delta = \varepsilon^2$.

$|x-y| = |\sqrt{x} - \sqrt{y}||\sqrt{x} + \sqrt{y}| \geq |\sqrt{x} - \sqrt{y}|^2$ $\forall x,y \in (0,\infty)$. Hence

$|f(x) - f(y)| = |\sqrt{x} - \sqrt{y}| < \varepsilon$ whenever $|x-y| < \delta = \varepsilon^2$. Thus f is uniformly continuous on $(0,\infty)$. Suppose f is Lipschitz on $(0,\infty)$, then $\exists K > 0$: $|f(x) - f(y)| \leq K|x-y|$ $\forall x,y \in (0,\infty)$. $|\sqrt{x} - \sqrt{y}| \leq K|x-y|$

So $\frac{1}{\sqrt{x}+\sqrt{y}} \leq K$ $\forall x,y \in (0,\infty)$. Choose x,y sufficiently small so we have a contadiction $\frac{1}{\sqrt{x}+\sqrt{y}} > K$. Thus f is not Lipschitz on $(0,\infty)$.

Let $a > 0$. $\frac{1}{\sqrt{x}+\sqrt{y}} < \frac{1}{2\sqrt{a}}$ $\forall x,y \in (a,\infty)$, hence $|\sqrt{x} - \sqrt{y}| \leq \frac{1}{2\sqrt{a}}|x-y|$ $\forall x,y \in (a,\infty)$. Thus f is Lipschitz on (a,∞).

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⑥ Put $M = \sup_{x\in\mathbb{R}} |\varphi'(x)|$. Sup exists as φ' is bounded. Let $x,y \in \mathbb{R}$ by MVT, $\exists c \in [x,y]$ s.t. $|\varphi(x) - \varphi(y)| = |\varphi'(c)||x-y| \leq M|x-y|$. Thus φ is Lipschitz. If $\sup_{x\in\mathbb{R}} |\varphi'(x)| < 1$ then $|\varphi(x) - \varphi(y)| < M|x-y|$ and $M < 1$, so φ is a contraction.

⑧ Define $T: C[0,1] \longrightarrow C[0,1]$. $T(y(x)) = \int_0^x F(z, y(z))dz$.

- We show that T is a contraction.

Since $F(x,y)$ is continuous and differentiable with respect to the second variable, by MVT, $|F(x,y_1) - F(x,y_2)| \leq |F_2(y_0)||y_1 - y_2| < \frac{1}{2}|y_1 - y_2|$ $\forall y_1, y_2 \in \mathbb{R}$. Consider

$$|T(y_1) - T(y_2)| = \left| \int_0^x F(z, y_1(z)) - F(z, y_2(z))\, dz \right| \leq \int_0^x |F(z,y_1) - F(z,y_2)| dz$$

$$\leq \frac{1}{2} \int_0^x |y_1(z) - y_2(z)| dz \underset{x\in[0,1]}{\leq} \frac{1}{2} \int_0^1 |y_1(z) - y_2(z)| dz = \frac{1}{2}|y_1 - y_2|.$$

Thus T is a contraction. Therefore T has a unique fixed point $f \in C[0,1]$. $T(f) = f$. so $\int_0^x F(z, f(z))dz = f(x)$ i.e $f' = F(x, f(x))$ and $f(0) = 0$. Thus the initial value problem has a unique solution.

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⑨ Consider $\varphi : (0,1) \longrightarrow (0,1)$ given by $\varphi(x) = \frac{x+1}{2}$. Then $|\varphi(x) - \varphi(y)| = \left|\frac{x+1}{2} - \frac{y+1}{2}\right| = \frac{|x-y|}{2} < |x-y|$. Suppose φ has a fixed point x' then $\varphi(x') = x'$ i.e. $\frac{x'+1}{2} = x'$ so $x' = 1 \notin (0,1)$, contradiction.

Thus φ has no fixed point.

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⑪ Let $(X_1, d_1), \ldots, (X_n, d_n)$ be separable metric spaces, and $A_1, \ldots, A_n$ be countable dense subsets of $X_1, \ldots, X_n$, respectively. Then $\overline{A_1 \times \cdots \times A_n} = \bar{A}_1 \times \cdots \times \bar{A}_n = X_1 \times \cdots \times X_n$ as the product is finite. Hence $A_1 \times \cdots \times A_n$ is dense in $X_1 \times \cdots \times X_n$. As A_i is countable for each i, $A_1 \times \cdots \times A_n$ is countable. Thus $X_1 \times \cdots \times X_n$ is separable.