Math 302

HW1

Solutions.

1- $s_n = \sum_{k=0}^{n} ar^k = a + ar + \cdots + ar^n = a(1+r+\cdots+r^n)$.

Let $T = 1+r+\cdots+r^n$. Then $Tr = r+r^2+\cdots+r^{n+1}$ and so

$Tr - T = (r+r^2+\cdots+r^{n+1}) - (1+r+\cdots+r^n) = r^{n+1}-1$. Hence

$T = \frac{1-r^{n+1}}{1-r}$. Thus $s_n = a\,\frac{1-r^{n+1}}{1-r}$.

2- We show that $\sum_{n=0}^{\infty} \frac{1}{n!}$ is convergent. The partial sum

$$s_n = \sum_{k=0}^{n} \frac{1}{k!} = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \cdots + \frac{1}{1.2.\cdots n}$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} < 1 + \sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{1-\frac{1}{2}}$$

$$= 3$$

Now $(s_n)_{n\in\mathbb{N}}$ is monotone and increasing, so it is convergent.

i.e. $\sum_{n=0}^{\infty} \frac{1}{n!}$ is convergent. From above we also have $2 < \sum_{n=0}^{\infty} \frac{1}{n!} < 3$.

Denote $\sum_{n=0}^{\infty} \frac{1}{n!} = e$. We show that it is irrational.

Observe that $e - s_n = \sum_{k=n+1}^{\infty} \frac{1}{k!} < \frac{1}{(n+1)!}\left(1 + \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots\right) = \frac{1}{n!\,n}$

Hence (*) $0 < e - s_n < \frac{1}{n!\,n}$. Suppose e is irrational, then $e = p/q$ for some positive integers p and q. Then by (*) $0 < q!(e-s_q) < \frac{1}{q}$

By assumption $p!e \in \mathbb{Z}$ and $q!s_q = q!\sum_{k=0}^{q} \frac{1}{k!} \in \mathbb{Z}$. Hence

$q!(e-s_q) \in \mathbb{Z}$. Since $q \geq 1$ and $0 < q!(e-s_q) < \frac{1}{q}$, $q!(e-s_q)$ is an integer between 0 and 1, contradiction.

Therefore $\sum_{k=0}^{\infty} \frac{1}{k!}$ is irrational.

5- Since $f(x)=\frac{1}{x}$ is continuous, decreasing on $[1,+\infty[$ we can use Integral test.

$$\int_1^\infty f(x)\,dx = \int_1^\infty \frac{1}{x}\,dx = \lim_{a\to\infty}\int_1^a \frac{1}{x}\,dx = \lim_{a\to\infty} \ln a = \infty.$$

Hence $\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent.

8- a) Let $p>0$. Given $\varepsilon>0$ choose $N=(1/\varepsilon)^{1/p}$. Then for $n\geq N$ $\frac{1}{n^p}<\frac{1}{N^p}=\varepsilon$. Hence $\lim_{n\to\infty}\frac{1}{n^p}=0$.

b) Let $p>0$. Put $\alpha_n=\sqrt[n]{p}-1$. If $p>1$, then $x_n>0$ and by binomial theorem $1+nx_n\leq(1+x_n)^n=p$, so $0<x_n\leq\frac{p-1}{n}$.

Hence $x_n\longrightarrow 0$ as $n\longrightarrow\infty$ i.e. $\lim_{n\to\infty}\sqrt[n]{p}=1$.

If $p=1$, then $\sqrt[n]{p}=1$ $\forall n\in\mathbb{N}$, so nothing to prove.

If $0<p<1$, then $1/p>1$ so from above $\lim_{n\to\infty}\sqrt[n]{1/p}=1$.

By properties of limit, $\lim_{n\to\infty}\sqrt[n]{p}=1$.

c) Put $\alpha_n=\sqrt[n]{n}-1$. Then $x_n\geq 0$ and by binomial theorem

$$n=(\alpha_n+1)^n\geq\frac{n(n-1)}{2}\alpha_n^2.\ \text{Hence}\quad 0\leq\alpha_n\leq\sqrt{\frac{2}{n-1}}\quad\text{for } n\geq 2.$$

Thus $\alpha_n\longrightarrow 0$ as $n\longrightarrow\infty$ i.e. $\lim_{n\to\infty}\sqrt[n]{n}=1$.

d) Let $p>0$ and $\alpha\in\mathbb{R}$. Let k be an integer s.t. $k>\alpha$, $k>0$.

For $n>2k$, $(1+p)^n>\binom{n}{k}p^k=\frac{n(n-1)\cdots(n-k+1)}{k!}p^k \underset{n>2k}{>}\frac{n^k p^k}{2^k k!}$

Hence $0<\frac{n^\alpha}{(1+p)^n}<\frac{2^k k!}{p^k}n^{\alpha-k}$. Since $\alpha-k<0$

$n^{\alpha-k}\longrightarrow 0$ as $n\longrightarrow\infty$, so $\lim_{n\to\infty}\frac{n^\alpha}{(1+p)^n}=0$.

e) Take $\alpha = 0$ in (d) so if $|x|<1$, then $|x|^n = \frac{1}{(1+p)^n}$ for some $p>0$. Result follows from (d).

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12 - Suppose $\sum |a_{n+1} - a_n|$ converges. Hence the partial sum sequence is cauchy. Hence given $\varepsilon > 0$, $\exists N \in \mathbb{N} : \forall m > n \geq N$ $\sum_{k=n}^{m} |a_{k+1} - a_k| < \varepsilon$. Therefore whenever $m > n \geq N$ we have

$$|a_m - a_n| = |a_m - a_{m-1} + a_{m-1} - \dots - a_{n+1} + a_{n+1} - a_n| = \left| \sum_{k=n}^{m-1} (a_{k+1} - a_k) \right|$$

$$\leq \sum_{k=n}^{m-1} |a_{k+1} - a_k| < \varepsilon.$$

Hence $(a_n)_{n \in \mathbb{N}}$ is cauchy and so it is convergent.

The convers is not true. Consider the sequence $a_n = \frac{(-1)^{n+1}}{n}$ $n \geq 1$. Clearly $(a_n)_{n \in \mathbb{N}}$ is convergent. But $|a_{n+1} - a_n| = \left| \frac{(-1)^{n+2}}{n+1} - \frac{(-1)^{n+1}}{n} \right|$ $= \frac{1}{n+1} + \frac{1}{n}$ so $\sum |a_{n+1} - a_n| = \sum \frac{1}{n+1} + \frac{1}{n}$ is divergent.