2) (5 pts each) For each of (a)-(d) below: If the proposition is true, write TRUE. If the proposition is false, write FALSE. No explanations are required for this problem.

2a) If $f, g \in R([a,b])$, then $f \cdot g \in R([a,b])$. ($R([a,b])$ is the space of Riemann integrable functions on $[a,b]$)

TRUE

2b) Let (f_n) be a sequence of uniformly continuous functions in $C([0,1])$. The pointwise limit of (f_n) is uniformly continuous on $[0,1]$.

FALSE. $(f_n(x) = x^n)$

2c) If $\mathcal{F}$ is an equicontinuous family in $C(\mathbb{R})$, then any $f \in \mathcal{F}$ is uniformly continuous on $\mathbb{R}$.

TRUE.

2d) Let f be a nonnegative function on $[a,b]$. Then, $f(x) = 0$ for every $x \in [a,b]$ if and only if $\int_a^b f(x)\,dx = 0$.

FALSE.

$$f(x) = \begin{cases} 0 & [0,1) \\ 1 & x = 1 \end{cases}$$

3a) (10 pts) Study the convergence (pointwise or uniform) of the following sequence

$$f_n(x) = \frac{x^n}{2^n},\ x \in [-1, 1]$$

UNIFORM: $f_n(x) \to 0$

$\varepsilon_0 > 0$ given. Let N_0 be s.t. $\frac{1}{2^{N_0}} < \varepsilon_0$

$\Rightarrow \forall n \geq N_0 \quad \left| \frac{x^n}{2^n} - 0 \right| = \frac{|x|^n}{2^n} < \frac{1}{2^n} < \varepsilon_0$ ✓

3b) (10 pts) Study the convergence (pointwise or uniform) of the series

$\sum_{n=1}^{\infty} x^n$ on $[-a, a]$, where $0 < a < 1$.

$$S_N(x) = \sum_{n=1}^{N} x^n = \frac{1-x^{N+1}}{1-x}$$

$S_N(x) \to \frac{1}{1-x}$ uniformly on $[-a,a]$

$\varepsilon_0 > 0$ given. $\left| S_N(x) - \frac{1}{1-x} \right| = \frac{|x|^{N+1}}{1-x} \leq \frac{|x|^{N+1}}{1-a} \leq \frac{a^{N+1}}{1-a} \quad \forall x \in [-a,a]$

let N_0 be s.t. $\frac{a^{N+1}}{1-a} < \varepsilon_0$ (since $0<a<1$, such an N_0 exits)

$\Rightarrow \forall n \geq N_0 \quad \left| S_n(x) - \frac{1}{1-x} \right| \leq \frac{x^{n+1}}{1-x} \leq \frac{|x|^{N+1}}{1-x} \leq \frac{a^{N+1}}{1-a} < \varepsilon_0$

D

4) (15 pts) Prove or give a counterexample for the following statement.

Let $\mathcal{A}$ be a unital subalgebra in $C([0,1])$. If $\mathcal{A}$ does not separate the points on $[0,1]$, then $\mathcal{A}$ cannot be dense in $C([0,1])$.

Assume A cannot sepate $a, b \in [0,1]$.
i.e. $\forall f \in A \quad f(a) = f(b) = C$

Let $g(x) = x$. Let $\varepsilon_0 = \left|\frac{b-a}{3}\right|$

Claim: $B_{\varepsilon_0}(g) \cap A = \emptyset$ (Hence A is not dense.)

$\forall f \in A \qquad d(f,g) = \sup_{[0,1]} |f(x) - g(x)| \geq \max\{|f(a) - g(a)|, |f(b) - g(b)|\}$

$= \max\{|C-a|, |C-b|\} \geq \frac{|a-b|}{2} > \varepsilon_0$

$\Rightarrow A \cap B_{\varepsilon_0}(g) = \emptyset$

D.

5) (15 pts) Let $f \in C([0,1])$. Prove that if $\int_0^1 x^n f(x)\,dx = 0$ for every $n \in \mathbb{N}$, then $f = 0$.

HW.9

6a) (10 pts) Let $\mathcal{F} = \{f_n(x) = x^n \mid x \in [0,1]\}$ be a family in $C([0,1])$.

Determine whether $\mathcal{F}$ is equicontinuous in $C([0,1])$ or not.

$\mathcal{F}$ is NOT EQUICONTINUOUS.

$\mathcal{F}$ is bdd as $\mathcal{F} \subseteq B_1(0) \subseteq C([0,1])$. $\left(\underset{\forall f \in \mathcal{F}}{\forall x \in [0,1]} \quad |f(x)| \leq 1 \right)$

Assume $\mathcal{F}$ is equicts. Then, $\bar{\mathcal{F}}$ is also equicts. ("Notes")

Hence, $\bar{\mathcal{F}}$ is closed, bdd, equicts $\underset{\text{Arzela-Ascoli}}{\Longrightarrow}$ $\bar{\mathcal{F}}$ is compact.

$\{f_n(x)\} \subset \bar{\mathcal{F}}$ is a seq. in $\bar{\mathcal{F}} \Rightarrow \exists$ convergent subsequence $f_{n_k} \to f$

However, $f_n \to g = \begin{cases} 0 & x \in [0,1) \\ 1 & x = 1 \end{cases}$ pointwise.

$\Rightarrow f_{n_k} \to g$. However, this convergence cannot be uniform as g is not cts

$\times$

b) (10 pts) Determine whether the closed unit ball $\mathbf{B}_1(0)$ in $C([0,1])$ is compact or not. ($\mathbf{B}_1(0) = \{f \in C([0,1]) \mid |f(x)| \leq 1, x \in [0,1]\}$)

$\bar{B}_1(0)$ not compact.

If $\bar{B}_1(0)$ was compact, any sequence $(f_n) \in \bar{B}_1(0)$ has a convergent subseq.

However, $f_n(x) = x^n$ has no convergent subseq. in $C([0,1])$.

$\Rightarrow \bar{B}_1(0)$ is not compact.