Math 302 - Problem Set # 10 - Spring 2011

Homework Problems: 2, 9, 11, 19, 20.

Let (X, d) be a complete metric space.

1. Prove that any subset of X that contains a second category set is of second category itself.
2. Prove that the complement of a first category subset of X is dense in X.
3. Prove that $C(X)$ with the supremum metric is of the second category in itself.
4. Prove that the union of countably many first category sets in a metric space is also of the first category.
5. Is $\mathbb{N}$ of the first category in $\mathbb{R}$? Is it of the first category in itself?
6. Prove that any second category set in $\mathbb{R}$ is uncountable.
7. Is $\mathbb{Q}$ of the first category in $\mathbb{R}$?
8. Is $\mathbb{R} \setminus \mathbb{Q}$ of the first category in $\mathbb{R}$? Is it a residual set?
9. Consider $C([0, 1])$ with the supremum metric. Prove that the set of polynomials on $[0, 1]$ is of the first category in $C([0, 1])$ and hence its complement is dense in $C([0, 1])$.
10. Consider $C([0, 1])$ with the supremum metric. Prove that the set of functions which are differentiable at least at one point of $(0, 1)$ is of the first category in $C([0, 1])$. Deduce that the set of nowhere differentiable functions is of the second category in $C([0, 1])$.
11. Give an example of a second category set in a metric space whose complement is also of the second category.
12. Prove that the intersection of two dense G_δ-sets in X is also a dense G_δ-set in X.
13. Prove that any dense G_δ-set in X is of the second category in X.
14. Suppose that (X, d) is complete. Prove that the complement of a dense G_δ-set in X is of the first category in X.
15. Is $\mathbb{Q}$ a G_δ-set in $\mathbb{R}$?
16. Prove that a bounded function f on X is continuous at a point $x_0 \in X$ iff the oscillation $O(f, x_0)$ of f at x_0 is 0.
17. Suppose that (X, d) is complete and let f be a Baire-1 function. Prove that D_f is of the first category in X.

18. Is the characteristic function $\chi_{\mathbb{Q}}$ of $\mathbb{Q}$ on $\mathbb{R}$ a Baire-1 function? Is it a Baire-2 function?

19. Give an example of a function f on $\mathbb{R}$ such that $C_f = \mathbb{R} \setminus \mathbb{Q}$.

20. Suppose that (X, d) is complete and let (f_n) be a sequence of continuous functions on X converging pointwise to f. Prove that there exists a nonempty open subset of X on which f is bounded.

21. Let (f_n) be a pointwise convergent sequence of continuous functions on $\mathbb{R}$. Prove that there exists a closed interval on which (f_n) is uniformly bounded.