Math 302 - Problem Set # 5 - Spring 2011

Homework Problems: 3, 5, 8, 9, 10.

1. Prove that any function of bounded variation on $[a, b]$ is R-integrable on $[a, b]$.

2. Let $t_0 \in [a, b]$ and $f : [a, b] \to \mathbb{R}$ be defined by $f(t_0) = 1$ and $f(x) = 0$ for every $x \in [a, b] \setminus \{t_0\}$. Prove that f is R-integrable and $\int_a^b f(x)\, dx = 0$.

3. Let f be a continuous, nonnegative function on $[a, b]$ with $\int_a^b f(x)\, dx = 0$. Prove that $f(x) = 0$ for every $x \in [a, b]$.

4. Let f be a continuous function on $[a, b]$. Prove that $f(x) = 0$ for every $x \in [a, b]$ if and only if $\int_c^d f(x)\, dx = 0$ for every $c, d \in [a, b]$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 0$ if $x \in \mathbb{Q}$ and $f(x) = 1$ if $x \notin \mathbb{Q}$. Prove that f is not R-integrable on any interval $[a, b]$ with $a < b$.

6. Let $f : [a, b] \to \mathbb{R}$ be a bounded function with the property that given any $\epsilon > 0$, the set $\mathfrak{D}_f$ of all points of discontinuity of f can be covered by finitely many intervals whose total length is less than ϵ. Prove that f is R-integrable on $[a, b]$.

7. Let f be a continuous function on $[a, b]$. For each positive integer n let $\mathfrak{P}$ be the partition of $[a, b]$ into n subintervals of equal length, σ_n be the Riemann sum of f with respect to $\mathfrak{P}$ with the choice of ξ_i as the left endpoint of the i^{th} subinterval, and Σ_n be the Riemann sum of f with respect to $\mathfrak{P}$ with the choice of ξ_i as the right endpoint of the i^{th} subinterval. Prove that $\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} \Sigma_n = \int_a^b f(x)\, dx$.

8. Evaluate $\int_0^1 x\, dx$ first by definition of the Riemann integral and then by using the previous problem.

9. Let f be a continuous function on $[0, 1]$. Prove that $\lim_{n\to\infty} \int_0^1 f(x^n)\, dx = f(0)$.

10. Let $f : [0, 1] \to [0, 1]$ be a continuous bijection. Prove that f^{-1} is R-integrable and

$$\int_0^1 f(x)\, dx + \int_0^1 f^{-1}(x)\, dx = 1\ .$$