Math 302 - Problem Set # 7 - Spring 2011

Homework Problems: 1, 3, 6, 11.

1. Let (f_n) be a sequence of functions defined on $[-\pi/2, \pi/2]$ by $f_n(x) = \begin{cases} \frac{\sin^2 nx}{n \sin x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$.

   (a) Prove that each f_n is continuous and (f_n) converges to the constant 0 function pointwise.

   (b) Prove that (f_n) converges to the constant 0 function uniformly on $[a, \pi/2]$ for any $a \in (0, \pi/2)$.

   (c) Find $\lim_{n\to\infty} f_n(\pi/2n)$ and using this prove that (f_n) does not converge to the constant 0 function uniformly on $[-\pi/2, \pi/2]$.

2. Let (f_n) be a sequence of functions defined on $[0, 2\pi]$ by $f_n(x) = n^2 \sin(x/n^2)$.

   (a) Prove that the sequence (f_n') converges uniformly to a function g on $[0, 2\pi]$.

   (b) Prove that (f_n) converges uniformly to a function f on $[0, 2\pi]$ with $f' = g$.

3. Let (f_n) be a sequence of differentiable functions with $|f_n'| \leq 1$ on an interval $[a, b]$. Prove that (f_n) converges uniformly if it converges pointwise.

4. Let (f_n) be a sequence of functions defined on $\mathbb{R}$ by $f_n(x) = \frac{e^{-n^2x^2}}{n}$.

   (a) Prove that (f_n) converges uniformly on $\mathbb{R}$ to a function f.

   (b) Prove that (f_n') converges f' pointwise on $\mathbb{R}$ but not uniformly on $[-a, a]$ for any $a > 0$.

5. Let (f_n) be a sequence of functions defined on $[0, 1]$ by $f_n(x) = nx(1-x)^n$. Compare

$$\lim_{n\to\infty} \left(\int_0^1 f_n(x)\, dx \right) \quad \text{and} \quad \int_0^1 \left(\lim_{n\to\infty} f_n(x) \right)\, dx\ .$$

6. Let (f_n) be a sequence of functions defined on $[-1, 1]$ by $f_n(x) = \frac{x^{2n}}{1+x^{2n}}$. Compare

$$\lim_{n\to\infty} \left(\int_0^1 f_n(x)\, dx \right) \quad \text{and} \quad \int_0^1 \left(\lim_{n\to\infty} f_n(x) \right)\, dx\ .$$

7. Let (f_n) be a sequence of functions defined on $[0, \pi/2]$ by $f_n(x) = n \cos^n x \sin x$. Compare

$$\lim_{n\to\infty} \left(\int_0^{\pi/2} f_n(x)\, dx \right) \quad \text{and} \quad \int_0^{\pi/2} \left(\lim_{n\to\infty} f_n(x) \right)\, dx\ .$$

8. Let (f_n) be a sequence of functions defined on $[0,\infty)$ by $f_n(x) = nxe^{-nx}$. Compare

$$\lim_{n\to\infty}\lim_{b\to\infty}\int_0^b f_n(x)\,dx \quad \text{and} \quad \lim_{b\to\infty}\lim_{n\to\infty}\int_0^b f_n(x)\,dx\ .$$

9. Let (f_n) be a sequence of functions defined on $[0,\infty)$ by $f_n(x) = \frac{xe^{-x/n}}{n}$.

   (a) Find the pointwise limit of (f_n) on $[0,\infty)$.

   (b) Prove that (f_n) converges uniformly on $[0,b]$ for any $b>0$.

   (c) Compare

$$\lim_{n\to\infty}\lim_{b\to\infty}\int_0^b f_n(x)\,dx \quad \text{and} \quad \lim_{b\to\infty}\lim_{n\to\infty}\int_0^b f_n(x)\,dx\ .$$

10. For each of the following sequences of functions, study the convergence (pointwise or uniform) on the given sets.

    (a) $f_n(x) = x^n(1-x)^n,\ x\in[0,1]$.

    (b) $f_n(x) = \frac{1}{1+nx^2},\ x\in\mathbb{R}$.

    (c) $f_n(x) = \frac{(1+x)^n-1}{(1+x)^n+1},\ x\in\mathbb{R}\setminus\{-2\}$.

    (d) $f_n(x) = \begin{cases} n, & \text{if } 0\le x\le 1/n \\ 0, & \text{if } 1/n < x \le 1 \end{cases},\ x\in(0,1]$.

    (e) $f_n(x) = \frac{\sin nx}{1+n^2x},\ x\in\mathbb{R}$.

    (f) $f_n(x) = \begin{cases} 1, & \text{if } x\in\{\alpha(0),\alpha(1),\ldots,\alpha(n)\} \\ 0, & \text{otherwise} \end{cases},\ x\in[0,1]$, where $\alpha:\mathbb{N}\to[0,1]\cap\mathbb{Q}$ is a given bijection.

11. For each of the following sequences of functions, study the convergence (pointwise or uniform) on the given sets.

    (a) $f_n(x) = a_nx^2,\ x\in\mathbb{R}$, where (a_n) is a sequence of real numbers converging to 1.

    (b) $f_n(x) = \frac{nx}{1+n^2x^2},\ x\in[0,\infty)$.

    (c) $f_n(x) = \frac{nx}{1+n^2x^2},\ x\in[a,\infty)$, where a is a positive real number.

    (d) $f_n(x) = nx^re^{-nx},\ x\in[0,\infty)$, where r is a given real number in $(0,1]$.

    (e) $f_n(x) = nx^re^{-nx},\ x\in[a,\infty)$, where r is a given real number in $(0,1]$ and a is a positive real number.

    (f) $f_n(x) = \frac{x^n}{1+x^n},\ x\in[0,\infty)$.