Math 302 - Problem Set # 9 - Spring 2011

Homework Problems: 4, 5, 8, 12, 13, 14, 17.

Let (X, d) be a metric space, $B(X)$ and $C(X)$ be the spaces of bounded and continuous real-valued functions on X, respectively, with the supremum metric ρ.

1. Let F be a family of bounded functions on X. Prove that F is a uniformly bounded family iff it is bounded as a subset of $B(X)$.

2. Prove that the union of two equicontinuous families in $C(X)$ is also equicontinuous.

3. Suppose that X is compact and (f_n) is a uniformly convergent sequence of continuous functions on X. Prove that (f_n) is equicontinuous.

4. Prove that the closure of an equicontinuous subset of $C(X)$ is also equicontinuous.

5. Let (f_n) be an equicontinuous sequence of functions on X and f be the pointwise limit of this sequence. Prove that f is continuous.

6. Suppose that X is compact, A is a dense subset of X and (f_n) is an equicontinuous sequence in $C(X)$. Also suppose that (f_n) converges to a continuous function $f : X \to \mathbb{R}$ pointwise on A. Prove that (f_n) converges uniformly to f on X.

7. Suppose that X is compact, $x \in X$ and define $\delta_x : C(X) \to \mathbb{R}$ by $\delta_x(f) = f(x)$ for every $f \in C(X)$. Prove that δ_x is continuous.

8. Suppose that X is compact and $F \subseteq C(X)$ is pointwise bounded and equicontinuous. Prove that F is uniformly bounded.

9. Suppose that X is compact and (f_n) is a uniformly bounded equicontinuous sequence in $C(X)$. Prove that (f_n) has a uniformly convergent subsequence.

10. Let (f_n) be a pointwise convergent sequence of differentiable functions on $[0, 1]$ such that (f_n') is a uniformly bounded sequence in $B([0, 1])$. Prove that (f_n) is uniformly convergent.

11. Suppose that N is a positive integer and F is the family of polynomials of degree at most N with real coefficients of absolute value no more than 1, restricted to $[0, 1]$. Prove that F is compact.

    For the follwoing questions, let (X, d) be a compact metric space, $B(X)$ and $C(X)$ be the spaces of bounded and continuous real-valued functions on X, respectively, with the supremum metric ρ.

12. Let A be a subalgebra of $C(X)$. Show that if the interior of A is nonempty, then $A = C(X)$.

13. Let $f \in C([0,1])$. Prove that if $\int_0^1 x^n f(x)\,dx = 0$ for every $n \in \mathbb{N}$, then $f = 0$.

14. Let $C_*([0,2\pi]) = \{f \in C([0,2\pi]) : f(0) = f(2\pi)\}$ and

$$A = \{\sum_{k=0}^{n}(a_k \cos(kx) + b_k \sin(kx) : x \in [0,2\pi], n \in \mathbb{N}, a_k, b_k \in \mathbb{R}\} .$$

Prove that A is a subalgebra of $C_*([0,2\pi])$ and A is dense in $C_*([0,2\pi])$.

15. Let $f \in C_*([0,2\pi])$. Suppose that $\int_0^1 f(x)\sin(nx)\,dx = \int_0^1 f(x)\cos(nx)\,dx = 0$. Prove that $f = 0$.

16. Let $A = \{\sum_{k=1}^{n} f_k(x)g_k(y) : f_k, g_k \in C([0,1]), n \in \mathbb{N}\}$. Prove that A is dense in $C([0,1] \times [0,1])$.

17. Let $A = \{\sum_{k=0}^{n} a_k e^{n_k x} : x \in [0,1], n, n_k \in \mathbb{N}, a_k \in \mathbb{R}\}$. Prove that A is dense in $C([0,1])$.

18. Prove that $(C(X), \rho)$ is a separable metric space.

In the following problems do not use Weierstrass Approximation Theorem or Stone-Weierstrass Theorem as the goal is to give another proof of the Weierstrass Approximation Theorem.

19. Define a sequence of polynomials on $[0,1]$ by $P_0(x) = 0$ and $P_{n+1}(x) = P_n(x) + \frac{1}{2}(x - P_n^2(x))$ for $n \in \mathbb{N}$. Prove that
    (a) For every $x \in [0,1]$ $P_n(x) \leq P_{n+1}(x) \leq \sqrt{x}$.
    (b) $(P_n(x))$ uniformly converges to $\sqrt{x}$.

20. Prove that there is a sequence of polynomials which converges uniformly to $|x|$ on $[-1,1]$.

21. Let $f \in C([0,1])$, $\epsilon > 0$, and $0 \leq a < b \leq 1$.
    (a) Prove that $\sup\{f(x) : x \in [a,b]\} - \inf\{f(x) : x \in [a,b]\} = \sup\{|f(x) - f(y)| : x, y \in [a,b]\}$.
    (b) Prove that there is a partition $\mathfrak{P} = \{\xi_0, \ldots, \xi_n\}$ of $[a,b]$ and real constants α_i, β_i for each $i \in \{0, \ldots, n-1\}$ such that the function g on $[a,b]$ defined by $g(x) = \alpha_i x + \beta_i$ for every $x \in [\xi_i, \xi_{i+1}]$ satisfies $\rho(f,g) < \epsilon$.

22. Using the previous problems give a new proof of Weierstrass Approximation Theorem.