Math 302

HW 10 - Solutions

Let (X,d) be a complete metric space.

2 - Let A be a first category subset of X, i.e. $A=\bigcup_{k=1}^{\infty} A_k$ with $(\overline{A_k})^\circ=\emptyset$. $\forall k\in\mathbb{N}$. Let $O_k=(\overline{A_k})^c$. Then O_k is open and $\overline{O_k}=\overline{(\overline{A_k})^c}=(\mathring{\overline{A_k}})^c=(\emptyset)^c=X$ i.e. O_k is dense in X, $\forall k\in\mathbb{N}$.

By Baire Category Theorem, $\cap O_k$ is dense in X. We also have $A\cap\left(\bigcap_{k=1}^{\infty} O_k\right)=\emptyset$. Hence $\bigcap_{k=1}^{\infty} O_k\subseteq A^c$ so $\overline{A^c}=X$, i.e. A^c is dense in X.

9 - Consider $C([0,1])$ with supremum metric. Let $P([0,1])$ be the set of polynomials on $[0,1]$ and $P_n([0,1])=\{p\in P([0,1]) : \deg p\leq n\}$. Then $P([0,1])=\bigcup_{n\geq 0} P_n([0,1])$. Let us see that $P_n([0,1])$ is closed.

Claim: Let $n\geq 1$. Then there exists $c>0$ s.t. for every choice of $a_0,\dots,a_n$, we have $\|a_0+a_1x+\dots+a_nx^n\|\geq c(|a_0|+\dots+|a_n|)$ $(*)$.

Proof: Let $s=\sum_{i=0}^{n}|a_i|$. Hence $(*)$ is equivalent to $\frac{\|a_0+\dots+a_nx^n\|}{s}\geq c$ i.e. $(**)$ $\|b_0+\dots+b_nx^n\|\geq c$ where $b_i=a_i/s$ and $\sum_{i=0}^{n}|b_i|=1$.

Hence it suffices to prove that $(**)$ holds for every $b_0,\dots,b_n$ with $\sum_{i=0}^{n}|b_i|=1$.

Suppose this is false. Then there exists a sequence $(y_m)_{m\in\mathbb{N}}$ s.t.

$$y_m=b_0^{(m)}+\dots+b_n^{(m)}x^n,\quad \sum_{i=0}^{n}|b_i^{(m)}|=1 \quad\text{and}\quad \|y_m\|\longrightarrow 0 \text{ as } m\longrightarrow\infty.$$

Since $\sum_{i=0}^{n}|b_i^{(m)}|=1$, $|b_i^{(m)}|\leq 1$, Hence for each i, the sequence $(b_i^{(m)})_{m\in\mathbb{N}}$ is bounded. By Bolzano-Weierstrass theorem, $(b_1^{(m)})_{m\in\mathbb{N}}$ has a convergent subsequence. Let b_1 be the limit of this subsequence, and let $(y_{1,m})_{m\in\mathbb{N}}$

denote the corresponding subsequence of $(y_m)_{m\in\mathbb{N}}$. By the same argument, $(y_{1,m})$ has a subsequence $(y_{2,m})$ for which the corresponding subsequence of $(b_2^{(m)})_{m\in\mathbb{N}}$ converges. Let b_2 denote the limit. Continuing this process, after n steps we obtain a subsequence $(y_{n,m})_{m\in\mathbb{N}}$ of $(y_m)_{m\in\mathbb{N}}$ s.t. $y_{n,m} = \sum_{j=0}^{n} c_j^{(m)} x^j$ $\sum_{j=0}^{n} |c_j^{(m)}| = 1$ and $c_j^{(m)} \longrightarrow b_j$ as $m \longrightarrow \infty$. Hence $y_{n,m} \longrightarrow y = \sum_{i=0}^{n} b_i x^i$ where $\sum_{i=0}^{n} |b_i| = 1$, so not all b_i can be zero. Since $\{1, x, \dots, x^n\}$ is a linearly independent set, we have $y \neq 0$. But as $\|y_m\| \to 0$ and $(y_{n,m})$ is a subsequence of (y_m), $\|y_{n,m}\| \longrightarrow 0$ contrary to $\|y\| \neq 0$. Thus $\exists c > 0 : \|a_0 + \dots + a_n x^n\| \geq c(|a_0| + \dots + |a_n|)$ for every choice of $a_0, \dots, a_n$.

Now let $(y_m)_{m\in\mathbb{N}}$ be a sequence in $P_n([0,1])$ converging to y. Then (y_m) is Cauchy. Given $\varepsilon > 0$ $\exists N \in \mathbb{N}$: $\|y_m - y_r\| < \varepsilon$ $\forall m, r > N$. By the claim, for some $c > 0$, we have $\varepsilon > \|y_m - y_r\| = \|\sum_{i=0}^{n} (a_i^{(m)} - a_i^{(r)}) x^i\|$ $\geq c \sum_{i=0}^{n} |a_i^{(m)} - a_i^{(r)}|$. Hence $|a_i^{(m)} - a_i^{(r)}| < \varepsilon/c$ $\forall i \in \{0, \dots, n\}$. Hence $(a_i^{(m)})_{m\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$ so is convergent say to a_i. Put $y' = a_0 + a_1 x + \dots + a_n x^n$. Then $\|y_m - y'\| = \|\sum_{i=0}^{n} (a_i^{(m)} - a_i) x^i\|$ $\leq \sum_{i=0}^{n} |a_i^{(m)} - a_i| \|x^i\| \longrightarrow 0$. Hence $y_m \longrightarrow y'$ so $y = y' \in P_n([0,1])$. This shows that $P_n([0,1])$ is closed, i.e. $\overline{P_n([0,1])} = P_n([0,1])$.

Let us see that $(P_n([0,1]))^\circ = \emptyset$. Let $f = a_0 + \dots + a_n x^n \in P_n([0,1])$. Let $\varepsilon > 0$. Let $g(x) = a_0 + \dots + a_n x^n + \varepsilon/2\, x^{n+1}$. Then $\|f - g\| = \varepsilon/2 < \varepsilon$. Hence

$g \in B_\varepsilon(f)$ and $g \notin P_n([0,1])$. Thus f cannot be an interior point, i.e. $(P_n([0,1]))^\circ = \emptyset$. Thus $P_n([0,1])$ is nowhere dense

Hence $P([0,1])$ is of the first category in $C([0,1])$ and hence its complement is dense in $C([0,1])$.

11- Let $X=\mathbb{R}$ with usual metric. Let $M=(0,\infty)$. Then as $\mathring{M}\neq\emptyset$ M is of second category. As $(M^c)^\circ=(-\infty,0)\neq\emptyset$, M^c is also of second category.

19- Let $\{r_n : n\in\mathbb{N}\}$ be an enumeration of $\mathbb{Q}$. Define $f:\mathbb{R}\longrightarrow\mathbb{R}$ by $f(x)=\sum_{r_n<x}\frac{1}{2^n}$. Let $x\in\mathbb{Q}$, hence $x=r_m$ for some $m\in\mathbb{N}$.

For $y>x$, $f(y)=\sum_{r_n<y}\frac{1}{2^n}\geq\sum_{r_n<x}\frac{1}{2^n}+\frac{1}{2^m}=f(x)+\frac{1}{2^m}$.

Hence $f(y)-f(x)\geq 1/2^m$ $\forall y>x$. Thus $\lim_{\substack{y\to x\\ y>x}} f(y)-f(x)\neq 0$. i.e. f is not continuous at x.

Now let $x\in\mathbb{R}\setminus\mathbb{Q}$. Let $\varepsilon>0$. Then $\exists N\in\mathbb{N}$ s.t. $\sum_{k=n}^{\infty}\frac{1}{2^k}<\varepsilon$ $\forall n\geq N$. as $\sum_{k=0}^{\infty}\frac{1}{2^k}$ is convergent. Consider the set $\{r_1,\dots,r_{N-1}\}$. Then $\exists\,\delta>0$ s.t. $(x-\delta,x+\delta)\cap\{r_1,\dots,r_{N-1}\}=\emptyset$. If $q\in\mathbb{Q}$ with $q\in(x-\delta,x+\delta)$, then $q=r_m$ for some $m\geq N$. Now let $y\in(x-\delta,x+\delta)$. We may assume $y>x$. Then

$$|f(y)-f(x)|=\left|\sum_{r_n<y}\frac{1}{2^n}-\sum_{r_n<x}\frac{1}{2^n}\right|=\left|\sum_{x\leq r_n<y}\frac{1}{2^n}\right|\leq\sum_{k=N}^{\infty}\frac{1}{2^k}<\varepsilon.$$

Thus f is continuous at x. i.e. $C_f=\mathbb{R}\setminus\mathbb{Q}$.

20- Let (X,d) is complete and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of continuous functions on X converging pointwise to f.

For each integer $k \geq 1$, let $M_k = \{x \in X : |f_n(x)| \leq k \ \forall n \in \mathbb{N}\}$.

Then $M_k = \bigcap_{n\in\mathbb{N}} \{x \in X : |f_n(x)| \leq k\}$. Since f_n is continuous, $\{x \in X : |f_n(x)| \leq k\}$ is closed so M_k is closed. As $(f_n)_{n\in\mathbb{N}}$ is pointwise convergent, $(f_n(x))_{n\in\mathbb{N}}$ is bounded $\forall x \in X$. Thus $\bigcup_{k\geq 1} M_k = X$. By Baire Category Theorem $\mathring{M}_k \neq \emptyset$ for some $k \geq 1$. Let $U = \mathring{M}_k$. For $x \in U$, $|f_n(x)| \leq k$ $\forall n \in \mathbb{N}$. Thus $|f(x)| \leq k$ $\forall x \in U$. i.e. f is bounded on U.