Math 302
HW-2 Solutions.

1- Let $\sum a_n$ be a convergent nonnegative series. By Cauchy-Schwarz inequality $\left(\sum_{n=0}^{N} \sqrt{a_n}\cdot\frac{1}{n}\right)^2 \leq \left(\sum_{n=0}^{N} a_n\right)\left(\sum_{n=0}^{N}\frac{1}{n^2}\right)$ so $\sum_{n=0}^{N}\sqrt{a_n}\,\frac{1}{n} \leq \left(\sum_{n=0}^{N} a_n\right)^{1/2}\left(\sum_{n=0}^{N}\frac{1}{n^2}\right)^{1/2}$

Since $\sum a_n$ and $\sum \frac{1}{n^2}$ conveges, their partial sum sequences are bounded so by the above inequality partial sum sequence of $\sum \frac{\sqrt{a_n}}{n}$ is bounded so $\sum \frac{\sqrt{a_n}}{n}$ converges.

3- Let $\sum_{n=1}^{\infty} a_n$ be a positive convergent series. Given $\varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. $\left|\sum_{k=n}^{n+p} a_n\right| < \varepsilon$ (*) $\forall n \geq N$ $\forall p \in \mathbb{N}$. For $n > N$ we have

$$\frac{\sum_{k=1}^{n} k a_k}{n} = \frac{\sum_{k=1}^{N} k a_k + \sum_{k=N+1}^{n} k a_k}{n} = \frac{1}{n}\sum_{k=1}^{N} k a_k + \sum_{k=N+1}^{n} \frac{k}{n} a_k .$$

Now $\frac{1}{n}\sum_{k=1}^{N} k a_k \longrightarrow 0$ as $n \longrightarrow \infty$ and $\sum_{k=N+1}^{n}\frac{k}{n}a_k \underset{\frac{k}{n}<1}{\leq} \sum_{k=N+1}^{n} a_k < \varepsilon$ by (*). Thus $\lim_{n\to\infty}\frac{\sum_{k=1}^{n} k a_k}{n} = 0$.

8- (a) Let $x \in \mathbb{R}$. We show that $\sum_{n=0}^{\infty}\frac{|x|^n}{n!}$ is convergent.

$\lim_{n\to\infty}\frac{|x|^{n+1}/(n+1)!}{|x|^n/n!} = \lim_{n\to\infty}\frac{|x|}{n+1} = 0$ so by Ratio test $\sum \frac{|x|^n}{n!}$ converges

i.e. $\sum \frac{x^n}{n!}$ is absolutely convergent.

(b) Let $x, y \in \mathbb{R}$. $S(x)S(y) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\frac{x^k}{k!}\frac{y^{n-k}}{(n-k)!}\right) = \sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}x^k y^{n-k}\right)$

$= \sum_{n=0}^{\infty}\frac{1}{n!}(x+y)^n = S(x+y)$.

10- Let $\sum a_n$, $\sum b_n$ be two series s.t.

(a) $\sum |a_{n+1} - a_n|$ is convergent

(b) The partial sum sequence of $\sum b_n$ is bounded.

(c) $\lim a_n = 0$,

Let $B_n = \sum_{k=0}^{n} b_k$ and $S_n = \sum_{k=0}^{n} a_k b_k$. Then for $m > n$ we have

$$|S_m - S_n| = \left|\sum_{k=n+1}^{m} a_k b_k\right| = \left|\sum_{k=n+1}^{m} a_k (B_{k+1} - B_k)\right| = \Big| a_{n+1}(B_{n+2} - B_{n+1}) + \cdots$$

$$+ a_m (B_{m+1} - B_m)\Big| = \Big| + B_{n+1} a_{n+1} + B_{n+2}(a_{n+1} - a_{n+2}) + \cdots + B_m(a_{m-1} - a_m)$$

$$+ a_m B_{m+1}\Big| \leq |B_{n+1}||a_{n+1}| + \sum_{k=n+1}^{m-1} |B_{k+1}||a_k - a_{k+1}| + |B_{m+1}||a_m|$$

(b) $\Rightarrow$ $\exists M > 0$: $|B_n| < M$ $\forall n \in \mathbb{N}$ hence

$$< M\left(|a_{n+1}| + \sum_{k=n+1}^{m-1} |a_k - a_{k+1}| + |a_m|\right)$$

Given $\varepsilon > 0$. (a) $\Rightarrow$ $\exists N \in \mathbb{N}$ $\sum_{k=n}^{m} |a_{k+1} - a_k| < \varepsilon/2M$ $\forall m, n > N$

and (c) $\Rightarrow$ $\exists N' \in \mathbb{N}$: $|a_n| < \varepsilon/4M$ $\forall n \geq N'$. Choose $m, n > \max(N, N')$

so

$$< M\left(\frac{\varepsilon}{2M} + \frac{\varepsilon}{2M}\right) = \varepsilon.$$

Thus $\sum a_n b_n$ is convergent.

11- First we show that $\left(\sum_{n=0}^{\infty} |a_n b_n|\right)^2 \leqslant \left(\sum_{n=0}^{\infty} a_n^2\right)\left(\sum_{n=0}^{\infty} b_n^2\right)$.

$$0 \leqslant \sum_{i=1}^{n} \sum_{j=1}^{n} (|a_i b_j| - |a_j b_i|)^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} (a_i^2 b_j^2 - 2|a_j a_i b_i b_j| + a_j^2 b_i^2)$$

$$= \sum_{i=1}^{n} a_i^2 \sum_{j=1}^{n} b_j^2 + \sum_{j=1}^{n} a_j^2 \sum_{i=1}^{n} b_i^2 - 2 \sum_{i=1}^{n} |a_i b_i| \sum_{j=1}^{n} |a_j b_j|$$

$$= 2 \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 - 2\left(\sum_{i=1}^{n} |a_i b_i|\right)^2 \Rightarrow \left(\sum_{i=1}^{n} |a_i b_i|\right)^2 \leqslant \left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right)$$

Suppose $\sum a_n^2$ and $\sum b_n^2$ are convergent so their partial sum sequences are bounded hence by the above inequality, partial sum sequence of $\sum |a_n b_n|$ is bounded. As it is a positive series it converges,