Math 302

HW5 - Solutions

3- Let f be a continuous, nonnegative function on $[a,b]$ with $\int_a^b f(x)\,dx = 0$. Assume that $\exists x_0 \in (a,b)$: $f(x_0) > 0$. Let $0 < \varepsilon < f(x_0)$

As f is continuous $\exists \delta > 0$: $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$.

Then for $x \in (x_0 - \delta, x_0 + \delta)$, $f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$. Hence

$\int_a^b f(x)\,dx \geq \int_{x_0-\delta}^{x_0+\delta} f(x)\,dx \geq \int_{x_0-\delta}^{x_0+\delta} (f(x_0) - \varepsilon)\,dx = 2\delta(f(x_0) - \varepsilon) > 0$, contradicting $\int_a^b f(x)\,dx = 0$. Thus $f(x) = 0$ $\forall x \in (a,b)$. Similarly $f(a) = f(b) = 0$.

Hence $f(x) = 0$ $\forall x \in [a,b]$.

5- Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q}. \end{cases}$

Let $a, b \in \mathbb{R}$ with $a < b$. Let $P = \{x_0 = a, x_1, \ldots, x_n = b\}$ be any partition of $[a,b]$. Then $L(f,P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}) = 0$ as $m_i = \inf_{x_{i-1} < x < x_i} f(x) = 0$ for every i. $U(f,P) = \sum_{i=1}^{n} M_i (x_i - x_{i-1}) = \sum_{i=1}^{n} (x_i - x_{i-1})$ $= (b - a)$ as $M_i = \sup_{x_{i-1} < x < x_i} f(x) = 1$ for every i. Thus,

$\sup_P L(f,P) = 0 \neq 1 = \inf_P U(f,P)$. i.e. $f \notin R([a,b])$.

8 - • Let $f(x)=x$. For $n\in\mathbb{N}$, let $P_n=\{0,\frac{1}{n},\dots,\frac{n-1}{n},1\}$. Then

$$L(f,P_n)=\sum_{i=1}^{n} m_i(x_i-x_{i-1})=\sum_{i=1}^{n}\frac{i-1}{n}\cdot\frac{1}{n}=\frac{n-1}{2n}\leq \sup_P L(f,P).\quad \forall n\in\mathbb{N}.$$

Hence $\lim_{n\to\infty} L(f,P_n)=\frac{1}{2}\leq \sup_P L(f,P)$.

$$U(f,P)=\sum_{i=1}^{n} M_i(x_i-x_{i-1})=\sum_{i=1}^{n}\frac{i}{n}\frac{1}{n}=\frac{n+1}{2n}\geq \inf_P U(f,P)\quad \forall n\in\mathbb{N}.$$

Hence $\lim_{n\to\infty} U(f,P_n)=\frac{1}{2}\geq \inf_P U(f,P)$. Therefore we have

$$\frac{1}{2}\leq \sup_P L(f,P)\leq \inf_P U(f,P)\leq\frac{1}{2}\quad \text{So}\quad \sup_P L(f,P)=\inf_P U(f,P)=\frac{1}{2}$$

Thus $\int_0^1 x\,dx=\frac{1}{2}$.

• $P=\{0,\frac{1}{n},\dots,\frac{n-1}{n},1\}$. $\sigma_n=\sum_{i=1}^{n}\frac{i-1}{n}\cdot\frac{1}{n}=\frac{n-1}{2n}\longrightarrow\frac{1}{2}$.

$\Sigma_n=\sum_{i=1}^{n}\frac{i}{n}\cdot\frac{1}{n}=\frac{n+1}{2n}\longrightarrow\frac{1}{2}$ Hence $\int_0^1 f(x)\,dx=\lim_{n\to\infty}\sigma_n=\lim_{n\to\infty}\Sigma_n=\frac{1}{2}$.

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9 - Let f be a continuous function on $[0,1]$. Given $\varepsilon>0$. Since f is continuous and $[0,1]$ is closed, f is bounded i.e. $|f(x)|<M$ $\forall x\in[0,1]$ for some $M>0$. Let $0<k<\frac{\varepsilon}{2(M-f(0))}$ Then $\left|\int_0^1 f(x^n)\,dx-f(0)\right|=\left|\int_0^1 (f(x^n)-f(0))\,dx\right|$

$$=\left|\int_0^{1-k}(f(x^n)-f(0))\,dx+\int_{1-k}^{1}(f(x^n)-f(0))\,dx\right|\leq\int_0^{1-k}|f(x^n)-f(0)|\,dx+\int_{1-k}^{1}|f(x^n)-f(0)|\,dx.\quad(*)$$

Now $\int_{1-k}^{1}|f(x^n)-f(0)|\,dx\leq (M-f(0))\int_{1-k}^{1}dx=(M-f(0))\,k<\varepsilon/2$. Since f is continuous at 0, $\exists\delta>0$ s.t. $|x|<\delta\implies|f(x)-f(0)|<\varepsilon/2$.

For $x\in[0,1-k]$, $|x^n|<(1-k)^n$. Choose $N\in\mathbb{N}$ s.t. $(1-k)^n<\delta$ $\forall n\geq N$.

Then $|x^n|<\delta$ $\forall x\in[0,1-k]$, $\forall n\geq N$. Hence $|f(x^n)-f(0)|<\varepsilon/2$ $\forall n\geq N$ and $\forall x\in[0,1-k]$. Hence $\int_0^{1-k}|f(x^n)-f(0)|\,dx\leq\frac{\varepsilon}{2}(1-k)<\varepsilon/2$.

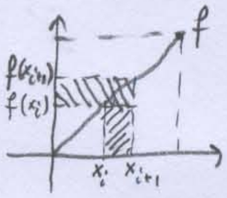
Hence by (*), for $n \geq N$, $\left|\int_0^1 f(x^n)dx - f(0)\right| < \varepsilon$

Thus $\int_0^1 f(x^n)\,dx \longrightarrow f(0)$ as $n \longrightarrow \infty$.

10- Let $f:[0,1] \longrightarrow [0,1]$ be a continuous bijection. Then f is monotone. Since f is bijective, f^{-1} is also bijective and hence f^{-1} is monotone. Thus f^{-1} is R-integrable. Let us see that $\int_0^1 f(x)dx + \int_0^1 f^{-1}(x)dx = 1$.

Suppose first f is increasing. Then $f(0)=0$ and $f(1)=1$. Let $P=\{x_0=0, x_1, \dots, x_n=1\}$ be a partition of $[0,1]$. Then $\{f(x_0), \dots, f(x_n)\}$ is also a partition of $[0,1]$, as f is bijective. Denote this partition as $f(P)$. Then

$$L(f,P) = \sum_{i=1}^{n} m_i (x_i - x_{i-1}) = \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}).$$



$$U(f^{-1}, f(P)) = \sum_{i=1}^{n} M_i (f(x_i) - f(x_{i-1})) = \sum_{i=1}^{n} f^{-1}(f(x_i))(f(x_i) - f(x_{i-1}))$$
$$= \sum_{i=1}^{n} x_i (f(x_i) - f(x_{i-1})).$$

We can write $1 = \sum_{i=1}^{n} (x_i f(x_i) - x_{i-1} f(x_{i-1}))$. Then we see that

$$1 - U(f^{-1}, f(P)) = \sum_{i=1}^{n} (x_i f(x_i) - x_{i-1} f(x_{i-1})) - \sum_{i=1}^{n} x_i (f(x_i) - f(x_{i-1})) = L(f,P).$$

Now $\int_0^1 f(x)dx = \sup_P L(f,P) = \sup_P (1 - U(f^{-1}, f(P))) = 1 - \inf_P U(f^{-1}, f(P))$.

Since f is bijective, every partition of $[0,1]$ is equal to $f(P)$ for some partition P. Hence $\inf_P U(f^{-1}, f(P)) = \inf_P U(f^{-1}, f(P)) = \int_0^1 f^{-1}(x)dx$

Thus $1 - \int_0^1 f^{-1}(x)dx = \int_0^1 f(x)dx$ i.e. $\int_0^1 f(x)dx + \int_0^1 f^{-1}(x)dx = 1$.

The case where f is decreasing is similar.