Math 302

HW-6 Solutions

2 - Let A be a nonempty bounded subset of $\mathbb{R}$. Then $\sup A$ and $\inf A$ exist. Let $\alpha = \sup A$ and $\beta = \inf A$. Let $\lambda = \sup\{a-b : a, b \in A\}$.

$a - b \leq \alpha - b \leq \alpha - \beta$, hence $\alpha - \beta$ is an upper bound of $\{a-b : a, b \in A\}$.

Let $\varepsilon > 0$. We show that $\exists a, b \in A$: $a - b > \alpha - \beta - \varepsilon$. Since $\alpha = \sup A$ $\exists a \in A$: $a > \alpha - \varepsilon/2$ and since $\beta = \inf A$ $\exists b \in A$: $b < \beta + \varepsilon/2$ hence $a - b > \alpha - \beta - \varepsilon$. This proves that $\alpha - \beta = \lambda$.

Let $\inf\{a-b : a, b \in A\} = \gamma$. Then $\sup\{|a-b| : a, b \in A\} = \begin{cases} \lambda & \text{if } \gamma \geq 0 \text{ or } |\gamma| < \lambda \\ -\gamma & \text{if } |\gamma| > \lambda \end{cases}$

To prove that $\sup\{|a-b| : a, b \in A\} = \alpha - \beta$, it is enough to prove that $-\inf\{a-b : a, b \in A\} = -\gamma = \alpha - \beta$.

$a - b \geq \beta - b \geq \beta - \alpha$ $\forall a, b \in A$ so $\beta - \alpha$ is a lower bound for $\{a-b : a, b \in A\}$. Let $\varepsilon > 0$. We show that $\exists a, b \in A$: $a - b < \beta - \alpha + \varepsilon$. Since $\beta = \inf A$, $\exists a \in A$: $a < \beta + \varepsilon/2$ and since $\alpha = \sup A$, $\exists b \in A$ s.t. $b > \alpha - \varepsilon/2$. Hence $a - b < \beta - \alpha + \varepsilon$. Thus $\beta - \alpha = \gamma$. i.e. $\alpha - \beta = -\gamma$. This proves $\alpha - \beta = -\inf\{a-b : a, b \in A\}$.

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4 - Let $f : [a, b] \longrightarrow \mathbb{R}$ be a bounded function.

(a) Suppose that f is R-integrable. Let $\varepsilon > 0$. Then there exists a partition P_ε s.t. $L(f, P_\varepsilon) > \int_a^b f(x)dx - \varepsilon$ and $U(f, P_\varepsilon) < \int_a^b f(x)dx + \varepsilon$.

Let P be a partition s.t. $P \supseteq P_\varepsilon$. Then $L(f, P) \geq L(f, P_\varepsilon)$ and $U(f, P) \leq U(f, P_\varepsilon)$ and as $L(f, P) \leq R(f, P) \leq U(f, P)$ we have $-\varepsilon < R(f, P) - \int_a^b f(x)dx < \varepsilon$. i.e. $\left| R(f, P) - \int_a^b f(x)dx \right| < \varepsilon$.

(b) Suppose there is a real number I s.t. for every $\varepsilon > 0$ there exists a partition P_ε of $[a,b]$ for which $|R(f,P) - I| < \varepsilon$ whenever P is a partition of $[a,b]$ with $P_\varepsilon \subseteq P$. Let $\varepsilon > 0$ and P_ε be a partition as above. Let $P \supseteq P_\varepsilon$. $R(f,P) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$, $\xi_i \in [x_{i-1}, x_i]$.

Let $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x_i)$. Hence $\exists \xi_i \in [x_{i-1}, x_i]$ s.t. $f(\xi_i) - m_i < \frac{\varepsilon}{b-a}$

Then $|L(f,P) - R(f,P)| \leq \sum_{i=1}^{n} (f(\xi_i) - m_i)(x_i - x_{i-1}) = \varepsilon$.

Let $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x_i)$. Hence $\exists \xi_i \in [x_{i-1}, x_i]$ s.t. $M_i - f(\xi_i) < \frac{\varepsilon}{b-a}$

so $|U(f,P) - R(f,P)| < \varepsilon$. Since $|R(f,P) - I| < \varepsilon$, we have

$|L(f,P) - I| \leq |L(f,P) - R(f,P)| + |R(f,P) - I| < 2\varepsilon$ and

$|U(f,P) - I| \leq |U(f,P) - R(f,P)| + |R(f,P) - I| < 2\varepsilon$. Thus

$|U(f,P) - L(f,P)| < 4\varepsilon$. Therefore f is R-integrable, and

$I = \int_a^b f(x)\,dx$.

7- Let $f: \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(0) = 0$ and $f(x) = \sin(1/x)$ if $x \neq 0$.

Since $|\sin(1/x)| \leq 1$, f is bounded on $[0,1]$. The only discontinuity of f is at $x = 0$. Hence $D_f = \{x \in [0,1] : f \text{ is discontinuous at } x\} = \{0\}$ so is negligable. Thus $f \in R([0,1])$.

11- Let $f \in R([a,b])$ and $c \in (a,b)$. Let $\varepsilon > 0$. Then as $f \in R([a,b])$ there exists a partition $P = \{x_0, \dots, x_n\}$ of $[a,b]$ s.t. $U(f,P) - L(f,P) < \varepsilon$. Let $P_1 = \{x_0, \dots, x_k, c\}$ and $P_2 = \{c, x_{k+1}, \dots, x_n\}$ be partitions of $[a,c]$ and $[c,b]$ respectively. Then $P' = P_1 \cup P_2 \supseteq P$ hence $L(f,P) \leq L(f,P')$ and $U(f,P') \leq U(f,P)$. Now,

$$L(f,P') = L(f,P_1) + L(f,P_2) \leq \int_a^c f(x)dx + \int_c^b f(x)dx \leq U(f,P_1) + U(f,P_2) = U(f,P')$$

i.e. $L(f,P) < \int_a^c f(x)dx + \int_c^b f(x)dx < U(f,P)$. We also have

$L(f,P) < \int_a^b f(x)dx < U(f,P)$. Thus $\left| \int_a^b f(x)dx - \left(\int_a^c f(x)dx + \int_c^b f(x)dx \right) \right| < \varepsilon$

As ε was arbitrary, this implies $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$.

12- Let $\zeta : (1,\infty) \longrightarrow \mathbb{R}$ be given by $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$.

(a) Let $s \in (1,\infty)$. Then by Integral test, $\int_1^{\infty} \frac{1}{x^s} dx = \frac{1}{-s+1} x^{-s+1} \Big|_1^{\infty} = \frac{1}{s-1}$

hence $\sum_{n=1}^{\infty} \frac{1}{n^s}$ is convergent $\forall s > 1$.

(b) Let $N \geq 1$ be an integer. Then $s\int_1^N \frac{[x]}{x^{s+1}} dx = s\sum_{k=1}^{N-1} \int_k^{k+1} \frac{[x]}{x^{s+1}} dx$, $\left(\begin{array}{l} [x] = k \\ \forall x \in [k,k+1) \end{array} \right)$

$$= s \sum_{k=1}^{N-1} \int_k^{k+1} \frac{k}{x^{s+1}} dx = s\sum_{k=1}^{N-1} k \left(-\frac{x^{-s}}{s} \right)\Big|_k^{k+1} = \sum_{k=1}^{N-1} k \left(\frac{1}{k^s} - \frac{1}{(k+1)^s} \right) = 1 - \frac{1}{2^s} + \frac{2}{2^s} - \frac{2}{3^s}$$

$$+ \frac{3}{3^s} - \frac{3}{4^s} + \dots + \frac{N-1}{(N-1)^s} - \frac{N-1}{N^s} = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \dots + \frac{1}{(N-1)^s} + \frac{1}{N^s} - \frac{1}{N^{s-1}}$$

$= \sum_{k=1}^{N} \frac{1}{k^s} - \frac{1}{N^{s-1}}$. Hence $s\int_1^{\infty} \frac{[x]}{x^{s+1}} dx = \lim_{N\to\infty} s \int_1^N \frac{[x]}{x^{s+1}} dx = \lim_{N\to\infty} \left(\sum_{k=1}^{N} \frac{1}{k^s} - \frac{1}{N^{s-1}} \right) = \zeta(s)$.

(c) $\frac{s}{s-1} - s\int_1^\infty \frac{x-[x]}{x^{s+1}}\,dx = \frac{s}{s-1} - s\int_1^\infty \frac{x}{x^{s+1}}\,dx + s\int_1^\infty \frac{[x]}{x^{s+1}}\,dx$

$$= \frac{s}{s-1} - s\int_1^\infty \frac{1}{x^s}\,dx + \zeta(s) = \frac{s}{s-1} - s\cdot\left(\frac{x^{-s+1}}{-s+1}\right)_{x=1}^{x=\infty} + \zeta(s)$$

$$\overset{(s>1)}{=} \frac{s}{s-1} + \frac{s}{1-s} + \zeta(s) = \zeta(s).$$