Math 302

HW9 - Solutions

4- Let H be an equicontinuous subset of $C(X)$. So we have

$\forall \varepsilon>0 \ \exists \delta>0 : |f(x)-f(y)|<\varepsilon$ whenever $d(x,y)<\delta$, $x,y\in X$ and $f\in H$.

Let $\varepsilon>0$. Then $\exists \delta>0$: $|f(x)-f(y)|<\varepsilon/3$ whenever $d(x,y)<\delta$, $x,y\in X$, $f\in H$

Let $f\in \bar{H}$. Then there exist sequences $(f_n)_{n\in\mathbb{N}}$ in H such that $f_n\to f$ i.e. $\sup_{x\in X}|f_n(x)-f(x)|\longrightarrow 0$ as $n\to\infty$.

Hence $\exists N\in\mathbb{N}$ s.t. $|f_n(x)-f(x)|<\varepsilon/3 \quad \forall n\geq N, \forall x\in X$.

Let $x,y\in X$ s.t. $d(x,y)<\delta$. Fix $n\geq N$. Then

$$|f(x)-f(y)|\leq \underbrace{|f(x)-f_n(x)|}_{<\varepsilon/3}+\underbrace{|f_n(x)-f_n(y)|}_{\varepsilon/3}+\underbrace{|f_n(y)-f(y)|}_{\varepsilon/3}$$

$$<\varepsilon.$$

Thus $\bar{H}$ is equicontinuous.

5- Let $(f_n)_{n\in\mathbb{N}}$ be an equicontinuous sequence of functions on X and f be the pointwise limit of $(f_n)_{n\in\mathbb{N}}$. Let $x_0\in X$. Let us see that f is continuous at x_0. Let $\varepsilon>0$. As $(f_n)_{n\in\mathbb{N}}$ is equicontinuous, $\exists\delta>0$ s.t. $|f_n(x)-f_n(y)|<\varepsilon/3$ whenever $d(x,y)<\delta$, $x,y\in X$, $n\in\mathbb{N}$.

As $f_n\to f$ pointwise, $\exists N_1\in\mathbb{N}$ s.t. $|f_n(x_0)-f(x_0)|<\varepsilon/3 \quad \forall n\geq N_1$.

Let $y\in X$ with $d(x_0,y)<\delta$. Since $f_n(y)\longrightarrow f(y)$, $\exists N_2\in\mathbb{N}$ s.t. $|f_n(y)-f(y)|<\varepsilon/3 \quad \forall n\geq N_2$. Let $N=\max\{N_1,N_2\}$. Then

$$|f(x_0)-f(y)|\leq |f(x_0)-f_N(x_0)|+|f_N(x_0)-f_N(y)|+|f_N(y)-f(y)|<\varepsilon/3+\varepsilon/3+\varepsilon/3=\varepsilon.$$

Thus f is continuous at x_0.

8- Suppose X is compact and $F \subseteq C(X)$ is pointwise bounded and equicontinuous. Let $\varepsilon > 0$. Then $\exists \delta > 0$ s.t. $|f(x)-f(y)| < \varepsilon$ whenever $d(x,y) < \delta$, $x,y \in X$, $f \in F$. Now $X \subseteq \bigcup_{x \in X} B_\delta(x)$. As X is compact, $\exists \{x_1, \dots, x_n\} \subseteq X$ s.t. $X \subseteq \bigcup_{i=1}^{n} B_\delta(x_i)$. Since F is pointwise bounded, for each i, $\exists M_i < \infty$ s.t. $|f(x_i)| < M_i$ $\forall f \in F$. Let $M = \max\{M_1, \dots, M_n\}$. Let $x \in X$. Then $x \in B_\delta(x_i)$ for some i. Let $f \in F$. Then $|f(x)| \leq |f(x) - f(x_i)| + |f(x_i)| \leq \varepsilon + M$

Thus $|f(x)| < \varepsilon + M$ $\forall f \in F$ and $\forall x \in X$. i.e. F is uniformly bounded

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12- Let A be a subalgebra of $C(X)$. Suppose $\mathring{A} \neq \emptyset$. Let $f \in \mathring{A}$. Then $\exists \delta > 0$ s.t. $B_\delta(f) = \{ g \in C(X) : \rho(f,g) < \delta \} \subseteq A$.

We show that $B_\delta(f) = \{f\} + B_\delta(0)$. Let $g \in B_\delta(f)$. Then $\rho(g-f, 0) = \sup_{x \in X} |(g-f)(x)| = \sup_{x \in X} |g(x) - f(x)| < \delta$ as $g \in B_\delta(f)$. Hence $B_\delta(f) \subseteq \{f\} + B_\delta(0)$. Let $h \in \{f\} + B_\delta(0)$. Then $h = f + g$ for some $g \in B_\delta(0)$. Hence $\rho(h,f) = \sup_{x \in X} |(f+g)(x) - f(x)| = \sup_{x \in X} |g(x)| < \delta$ as $g \in B_\delta(0)$. Hence $B_\delta(f) = \{f\} + B_\delta(0)$.

Let $g \in C(X)$. For every $\lambda \in \mathbb{R}$, $\rho(\lambda g, 0) = \sup_{x \in X} |\lambda g(x)| = |\lambda| \sup_{x \in X} |g(x)| = |\lambda| \rho(g,0)$. Choose $|\lambda|$ sufficiently small so that $\rho(\lambda g, 0) < \delta$. Hence $\lambda g \in B_\delta(0)$. Since $B_\delta(0) = B_\delta(f) - \{f\}$ and A is a subalgebra $B_\delta(0) \subseteq A$. Hence $\lambda g \in A$. Again as A is a subalgebra $\frac{1}{\lambda} \cdot \lambda g = g \in A$. Thus $C(X) = A$.

13- Let $f \in C([0,1])$. Suppose $\int_0^1 x^n f(x)\,dx = 0 \quad \forall n \in \mathbb{N}$.

Let $p(x)$ be a polynomial. Then $p(x) = a_n x^n + \dots + a_0$. Hence $\int_0^1 p(x)f(x)\,dx = \sum_{k=0}^{n} a_k \int_0^1 x^k f(x)\,dx = 0$ by assumption. Thus

$\int_0^1 p(x)f(x)\,dx = 0$ for every polynomial. Since $f \in C([0,1])$

by Stone-Weierstrass theorem, there exists a sequence $(P_n)_{n\in\mathbb{N}}$ of polynomials s.t. $P_n \to f$ uniformly on $[0,1]$. As $f \in C([0,1])$ f is bounded i.e. $\exists M < \infty : |f(x)| < M \quad \forall x \in [0,1]$. Then

$$\sup_{x\in[0,1]} |P_n(x)f(x) - f(x)^2| = \sup_{x\in[0,1]} |f(x)|\,|P_n(x) - f(x)| \leq M \sup_{x\in[0,1]} |P_n(x) - f(x)| \to 0$$

hence $P_n f \longrightarrow f^2$ uniformly. Thus $\lim_{n\to\infty} \int_0^1 P_n(x)f(x)\,dx = \int_0^1 f^2(x)\,dx$

and as $\int_0^1 P_n(x)f(x)\,dx = 0 \quad \forall n \in \mathbb{N}$, $\int_0^1 f^2(x)\,dx = 0$. Now f^2 is continuous nonnegative function on $[0,1]$ with $\int_0^1 f^2(x)\,dx$. Hence by PS5, question 3, $f(x)^2 = 0 \quad \forall x \in [0,1]$. i.e. $f = 0$.

14- Let $C_*([0,2\pi]) = \{ f \in C([0,2\pi]) : f(0) = f(2\pi) \}$, and

$A = \{ \sum_{k=0}^{n} (a_k \cos kx + b_k \sin kx) : x \in [0,2\pi],\ n \in \mathbb{N},\ a_k, b_k \in \mathbb{R} \}$.

Clearly, $A \subseteq C_*([0,2\pi])$. $\sum_{k=0}^{n_1}(a_k \cos kx + b_k \sin kx) + \sum_{k=0}^{n_2}(a_k' \cos kx + b_k' \sin kx)$

$= \sum_{k=0}^{\max\{n_1, n_2\}} [(a_k + a_k') \cos kx + (b_k + b_k') \sin kx] \in A$ i.e. closed under addition. Let $\lambda \in$

$\lambda \sum_{k=0}^{n}(a_k \cos kx + b_k \sin kx) = \sum_{k=0}^{n}(\lambda a_k \cos kx + \lambda b_k \sin kx) \in A$ i.e. closed under

scalar multiplication. $\sum_{k=0}^{n_1}(a_k \cos kx + b_k \sin kx) \cdot \sum_{j=0}^{n_2}(a_j' \cos jx + b_j' \sin jx)$

$= \sum_{k=0}^{n_1} \sum_{j=0}^{n_2} (a_k a_j' \cos kx \cos jx + a_k b_j' \cos kx \sin jx + b_k a_j' \sin kx \cos jx$

$+ b_k b_j' \sin kx \sin jx) \in A$ by using trigonometric identities.

Since $1 \in A$, A is a unital subalgebra of $C_*([0,2\pi])$.

Let $K = \{ z \in \mathbb{C} : |z| = 1 \}$ and consider $C(K)$. Let $\varphi : [0,2\pi] \to K$

$\varphi(x) = (\cos x, \sin x)$ Then $\varphi \in C([0,2\pi])$.

Let $T : C(K) \longrightarrow C_*([0,2\pi])$, $T(f) = f \circ \varphi$. Then T is continuous and onto. Let $A = \{ p(z) : p$ is a polynomial$\}$.

Then $A \subseteq C(K)$ and $\overline{A} = C(K)$, by Stone-Weierstrass theorem.

$T(A) = \{ p \circ \varphi : p$ is a polynomial$\} = \{ \sum_{k=0}^{n}(a_k \cos kx + b_k \sin kx)$

$a_k, b_k \in \mathbb{R},\ n \in \mathbb{N} \} = A_0$. Since $\overline{A} = C(K)$ and T is continuous and onto, $\overline{A_0} = C_*([0,2\pi])$.

17- Let $A = \{ \sum_{k=0}^{n} a_k e^{n_k x} ; x \in [0,1], n, n_k \in \mathbb{N}, a_k \in \mathbb{R} \}$.

Clearly, A is a unital subalgebra of $C([0,1])$. Let us see that A separates points of $[0,1]$. Let $x, y \in [0,1]$ with $x \neq y$. Then take $f(x) = e^x \in A$. Then $f(x) \neq f(y)$. Thus A separates points of $[0,1]$. By Stone-Weierstrass theorem $\overline{A} = C([0,1])$.