Math 402/571 Topology
Final Exam

January 4, 2016

1a) (4 pts) State Heine-Borel Theorem.
1b) (4 pts) Define homotopy equivalence of two spaces.
1c) (4 pts) Define triangulation of a surface.
1d) (4 pts) State the Classification Theorem for Compact Non-orientable Surfaces.

2) (4 pts each) For each of (a)-(d) below: If the proposition is true, write TRUE. If the proposition is false, write FALSE.
2a) Image of a closed set under a continuous function is closed.
2b) Let S be a compact orientable surface. If $\chi(S)$ is given, then $\pi_1(S)$ can be computed.
2c) Fundamental group of any compact surface with boundary is a free group.
2d) Fundamental group determines the orientability for closed surfaces.

3) Let X be a metric space, and $A \subset X$.
Prove or give a counterexample for the following statements:
3a) (8 pts) If A is compact, then A is closed and bounded.
3b) (7 pts) If A is closed and bounded, then A is compact.

4) (12 pts) Let Σ_g^k be a compact, orientable surface of genus g with k boundary components. Let N_3^2 be the surface obtained from non-orientable surface N_3 by removing 2 open disks.
Find all possible (g, k) pairs so that $\Sigma_g^k \sim N_3^2$ (homotopy equivalent).

5) Determine the following surfaces according to the Classification Theorem, i.e. Find corresponding Σ_g^k or N_q^k.
5a) (8 pts) $S = 2$ disks connected with 3 straight and 2 twisted strips.
5b) (8 pts) $T = \Sigma_2 \sharp N_4$

6a) (10 pts) Let $X = S^3 - \{3\, points\}$.
Is X simply connected? Is X contractible? Show your work.
6b) (10 pts) Let $X = \mathbb{R}^3 - \{(0,0,0)\}$. Consider the $\mathbb{Z}$ action on X as follows: For $n \in \mathbb{Z}$, let $\varphi_n(x, y, z) = 2^n(x, y, z)$. Find the orbit space $Y = X/\mathbb{Z}$.

7) (15 pts) Let $X = T^2 - \{p\}$, a torus with one point removed.
Let $Y = X \times S^1$. Find $\pi_1(Y)$ and $\chi(Y)$.

2) (4 pts each) For each of (a)-(d) below: If the proposition is true, write TRUE. If the proposition is false, write FALSE. No explanations are required for this problem.

2a) Image of a closed set under a continuous function is closed.

FALSE. Ex: $f : \mathbb{R} \to (-\pi/2, \pi/2)$
$\arctan x$

2b) Let S be a compact orientable surface. If $\chi(S)$ is given, then $\pi_1(S)$ can be computed.

FALSE.

$S = \Sigma_2$ $\quad \chi(S) = \chi(T) = -2$

$T = \Sigma_1^2$ $\quad \pi_1(S) = \langle a_1, b_1, a_2, b_2 \mid [a_1,b_1][a_2,b_2] \rangle$

$\pi_1(T) = \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$

2c) Fundamental group of any compact surface with boundary is a free group.

TRUE.

$\pi_1(\Sigma_g^k) = \underbrace{\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}}_{2g+k-1}$ $\qquad \Sigma_g^k \sim \underbrace{S^1 \vee S^1 \vee \cdots \vee S^1}_{2g+k-1}$

2d) Fundamental group determines the orientability for closed surfaces.

TRUE.

$\pi_1(\Sigma_g) = \langle a_1, b_1, \dots, a_g, b_g \mid [a_1,b_1][\dots]\cdots[a_g,b_g] \rangle$

$\not\cong$

$\pi_1(N_g) = \langle a_1, a_2, \dots, a_g \mid a_1^2 a_2^2 \cdots a_g^2 \rangle$

3) Let X be a metric space, and $A \subset X$.
Prove or give a counterexample for the following statements:

3a) (8 pts) If A is compact, then A is closed and bounded.

TRUE. A closed: We will show A^c open. X metric space $\Rightarrow$ Hausdorff

$x \in A^c \Rightarrow \forall a \in A \; \exists \; U_a, V_x^a$ s.t. $a \in U_a^x$ open, $x \in V_x^a$ open, $U_a \cap V_x^a = \emptyset$

Consider covering $F_x = \{U_a^x\}$ s.t. $A \subseteq U_x^a$

A compact $\Rightarrow A \subseteq \bigcup_{i=1}^{n} U_{a_i}^x \Rightarrow V_x = \bigcap_{i=1}^{n} V_x^{a_i}$ open and $V_x \cap A = \emptyset$

$\Rightarrow V_x \subseteq A^c \Rightarrow A^c$ open $\Rightarrow A$ closed.

A bounded. Fix $a_0 \in A$. Consider covering $F = \{B_n(a_i)\}$ $\Rightarrow$ n-ball

A compact $\Rightarrow \exists$ max N $A \subseteq B_N(a_i)$

3b) (7 pts) If A is closed and bounded, then A is compact.

FALSE: $X = [0,1]$ d discrete metric.

$A = X \Rightarrow A$ is closed.

$A \subseteq B_2(0) \Rightarrow$ bdd

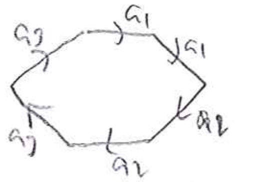
but A is not compact.

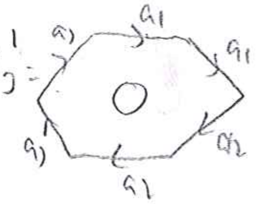
$F = \{ B_{1/2}(x) \mid x \in [0,1] \}$ has no finite covering

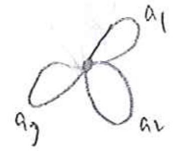
$B_{1/2}(x) = \{x\}$

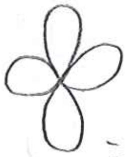
4) (12 pts) Let Σ_g^k be a compact, orientable surface of genus g with k boundary components. Let N_3^2 be the surface obtained from non-orientable surface N_3 by removing 2 open disks.

Find all possible (g, k) pairs so that $\Sigma_g^k \sim N_3^2$ (homotopy equivalent).

$N_3 =$ 

$N_3^1 =$ 

$\sim$ 

$\Rightarrow \quad N_3^2 \sim$  (new puncture brings new loop)

$= S^1 \vee S^1 \vee S^1 \vee S^1 \quad \Rightarrow \quad \chi(N_3^2) = \chi(\vee^4 S^1) = 1-4=-3$

$\chi(\Sigma_g^k) = 2-2g-k=-3$

$g=0 \Rightarrow k=5$ $\quad \Sigma_0^5 \sim$  ✓

$g=1 \Rightarrow k=3$ $\quad \Sigma_1^3 \sim$  ✓

$g=2 \Rightarrow k=1$ $\quad \Sigma_2^1 \sim$  ✓

$g \geq 3 \Rightarrow \chi(\Sigma_g^k) \leq -4 \Rightarrow$ pairs (g,k)

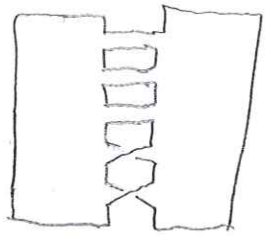
$(0,5)$

$(1,3)$ ✓

$(2,1)$

5) Determine the following surfaces according to the Classification Theorem, i.e. Find corresponding Σ_g^k or N_q^k.

5a) (8 pts) $S = 2$ disks connected with 3 straight and 2 twisted strips.



S

∂S: 3 components

S nonorientable

$\chi(S) = -3$

$\Rightarrow S \simeq N_2^3$

$S \sim$

$\chi(S) = 2 - 5 = -3$

5b) (8 pts) $T = \Sigma_2 \sharp N_4$

$T = \Sigma_2 \# N_4 = (\Sigma_2 - \text{disk}) \cup (N_4 - \text{disk})$

$\Rightarrow \quad \chi(T) = -3 + -3 + 0 = -6$

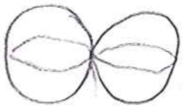
T is closed surface (no bdry)

T is not orientable ($\exists$ N_4 → Möbius band)

$\Rightarrow \quad T = N_8$

6a) (10 pts) Let $X = S^3 - \{3\ points\}$.
Is X simply connected? Is X contractible? Show your work.

$S^3 - \{1\ pt\} \simeq \mathbb{R}^3 \simeq$ unit open ball

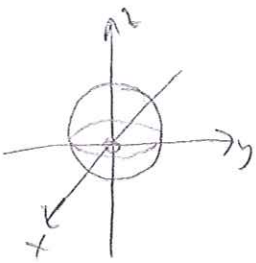
$\Rightarrow X \simeq$  $\simeq$  $= S^2 \vee S^2$

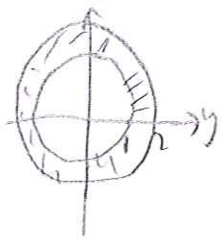
$\Rightarrow$ X simply conn. (Van Kampen)

but X is not contractible

$\chi(X) = 2+2-1 = 3 \neq 1 = \chi(pt)$

6b) (10 pts) Let $X = \mathbb{R}^3 - \{(0,0,0)\}$. Consider the $\mathbb{Z}$ action on X as follows: For $n \in \mathbb{Z}$, let $\varphi_n(x,y,z) = 2^n(x,y,z)$. Find the orbit space $Y = X/\mathbb{Z}$.





$\Omega = S^2 \times (1,2)$ fundamental domain.

since $\varphi_n(\Omega) = \Omega_n$

$\Rightarrow \bigcup \Omega_n = X$

$\Rightarrow Y = X/\mathbb{Z} = \Omega/\sim \; = \Omega/_{(x,1)\sim(x,2)} \Rightarrow Y = S^2 \times S^1$

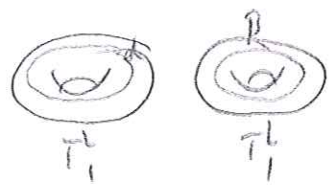
$x \in S^2$

7) (15 pts) Let $X = T^2 - \{p\}$, a torus with one point removed.

Let $Y = X \times S^1$. Find $\pi_1(Y)$ and $\chi(Y)$.

$X \sim \infty$

$Y = X \times S^1 \sim \infty \times S^1 =$



$T_1^1 \cup T_2^1 / \alpha \sim \beta$

$\Rightarrow \quad \chi(Y) = \chi(T_1^2) + \chi(T_2^1) - \chi(\alpha)$

$= 0 + 0 - 0 = 0$

$X = T^3 - (\{p\} \times S^1)$

$\pi_1(Y) = \pi_1(T_1^1) * \pi_1(T_2^1) / a_1 = a_2 = \langle a_1, b_1 \mid a_1 b_1 a_1^{-1} b_1^{-1} \rangle * \langle a_2, b_2 \mid a_2 b_2 a_2^{-1} b_2^{-1} \rangle / a_1 \sim a_2$

Van Kampen

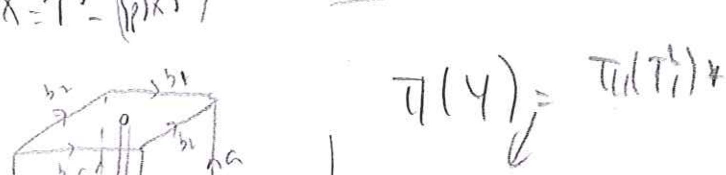
$= \langle a, b_1, b_2 \mid a b_1 a^{-1} b_1^{-1}, a b_2 a^{-1} b_2^{-1} \rangle$

$a \in$ center of $\pi_1(Y)$

b_1, b_2 does not commute

$\pi_1(Y)$ 3 generator: a, b_1, b_2

2 relations $[a, b_1]$
$[a, b_2]$



$\pi_1(Y) = \underset{b_1 \quad b_2}{(\mathbb{Z} * \mathbb{Z})} \times \underset{a}{\mathbb{Z}}$