Math 402/571 Topology

Midterm 2

December 4, 2015

1a) (5 pts) Define the fundamental group, $\pi_1(X, p)$.
1b) (5 pts) Define deformation retraction.
1c) (5 pts) Define group action on a space.
1d) (5 pts) State Jordan curve theorem.

2) (5 pts each) For each of (a)-(d) below: If the proposition is true, write TRUE. If the proposition is false, write FALSE. No explanations are required for this problem.

2a) Let X be <u>connected</u> and $p, q \in X$. Then $\pi_1(X, p) \simeq \pi_1(X, q)$.
2b) Any two simply connected spaces are homotopy equivalent.
2c) X and Y are contractible if and only if $X \times Y$ is contractible.
2d) Let $p \in X$ and $q \in Y$. If $X - \{p\}$ is homeomorphic to $Y - \{q\}$, then X is homotopy equivalent to Y.

3a) (10 pts) Show that S^2 and S^3 are not homeomorphic.

3b) (10 pts) Let T^2 be a torus, and $p, q \in T^2$ with $p \neq q$. Let X be the space obtained from T^2 by identifying p and q, i.e. $X = T^2/p \sim q$. Find $\pi_1(X)$.

4a) (10 pts) Let Σ_2 be genus 2 surface, and $p \in \Sigma_2$. Let $X = \Sigma_2 - \{p\}$.
Let Y be the sphere S^2 removed k points. If $X \sim Y$, find k.

4b) (15 pts) Let T^2 be the torus in $\mathbf{R}^3$ with rotation axis z-axis. Define $\mathbf{Z}_2$ action on T^2 with $\varphi(x, y, z) = (-x, -y, -z)$. Find the orbit space, and compute its fundamental group.

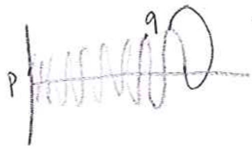
5) Prove or give a counterexample for the following statements:
5a) (10 pts) Let D be the open unit disk in $\mathbf{R}^2$. Then, any continuous map $f : D \to D$ has a fixed point, i.e. $\exists x \in D$ s.t. $f(x) = x$.

5b) (15 pts) If $X \cup Y$ is contractible, then X or Y is contractible.

2) (5 pts each) For each of (a)-(d) below: If the proposition is true, write TRUE. If the proposition is false, write FALSE. No explanations are required for this problem.

2a) Let X be connected and $p, q \in X$. Then $\pi_1(X,p) \simeq \pi_1(X,q)$.

FALSE. X connected $\not\Rightarrow$ X path connected.



X top. sine curve

$\pi_1(X,p) \neq \pi_1(X,q)$

2b) Any two simply connected spaces are homotopy equivalent.

FALSE. $X = pt$ $\quad \pi_1(X) = \pi_1(Y) = \{0\}$

$Y = S^2$ $\quad X \not\sim Y$

2c) X and Y are contractible if and only if $X \times Y$ is contractible.

TRUE.

2d) $X - \{p\} \simeq Y - \{q\} \quad \Rightarrow \quad X \sim Y$.

i.e. Let $p \in X$ and $q \in Y$. If $X - \{p\}$ is homeomorphic to $Y - \{q\}$, then X is homotopy equivalent to Y.

FALSE: $X = S^1$

$Y = [0,1)$

3a) (10 pts) Show that S^2 and S^3 are not homeomorphic.

Assume $\exists\, \varphi : S^2 \to S^3$ homeo.

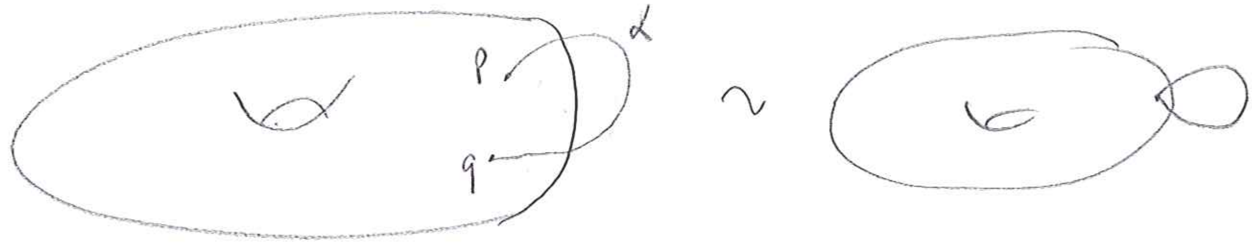
$p \in S^2 \quad q = \varphi(p) \quad \Rightarrow \quad \underset{\mathbb{R}^2}{\underset{\wr}{S^2 - \{p\}}} \simeq \underset{\mathbb{R}^3}{\underset{\wr}{S^3 - \{q\}}}$ ✗

$\mathbb{R}^2$ and $\mathbb{R}^3$ are not homeomorphic.

$\underset{S^1}{\underset{\wr}{\mathbb{R}^2 - \{p_1\}}} \not\simeq \underset{S^2}{\underset{\wr}{\mathbb{R}^3 - \{p_2\}}}$ since $\pi_1(\mathbb{R}^2 - \{p_1\}) = \mathbb{Z}$

$\pi_1(\mathbb{R}^3 - \{p_2\}) = (0)$

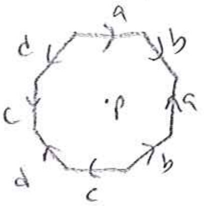
3b) (10 pts) Let T^2 be a torus, and $p, q \in T^2$ with $p \neq q$. Let X be the space obtained from T^2 by identifying p and q, i.e. $X = T^2/p \sim q$. Find $\pi_1(X)$.

$X \sim$ 

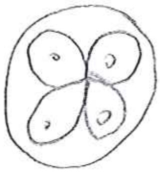
$\Rightarrow \quad X \sim T^2 \vee S^1 \quad \Rightarrow \quad \pi_1(X) = \pi_1(T^2) * \pi_1(S^1)$

$= (\mathbb{Z} \times \mathbb{Z}) * \mathbb{Z}$

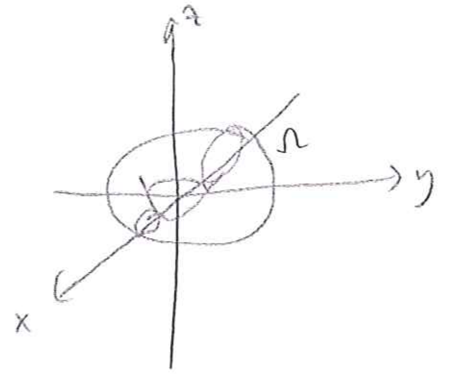
4a) (10 pts) Let Σ_2 be genus 2 surface, and $p \in \Sigma_2$. Let $X = \Sigma_2 - \{p\}$. Let Y be the sphere S^2 removed k points. If $X \sim Y$, find k.

$\Sigma_2 =$ 

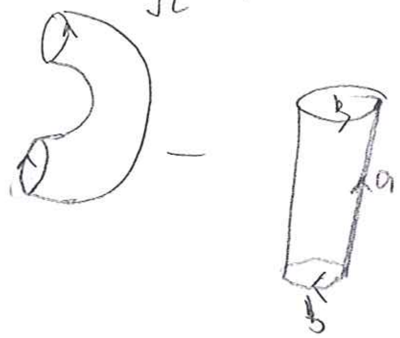
$\Rightarrow \Sigma_2 - \{p\} \sim S^1 \vee S^1 \vee S^1 \vee S^1$

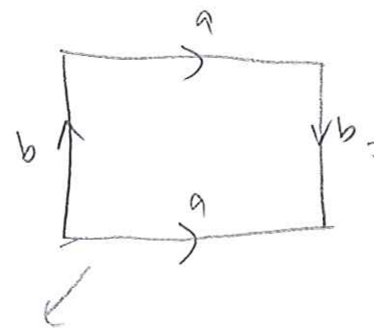
$S^2 - 5$ pts $\sim D^2 - 4$ pts $\longrightarrow$  $\Rightarrow$ $k = 5$

4b) (15 pts) Let T^2 be the torus in $\mathbf{R}^3$ with rotation axis z-axis. Define $\mathbf{Z}_2$ action on T^2 with $\varphi(x, y, z) = (-x, -y, -z)$. Find the orbit space, and compute its fundamental group.



$\Omega \to$ fundamental domain



$\Rightarrow$ 

$= K^2$ Klein bottle

$\pi_1(K^2) = \langle a, b \mid aba^{-1}b \rangle$

5) Prove or give a counterexample for the following statements:

5a) (10 pts) Let D be the open unit disk in $\mathbf{R}^2$. Then, any continuous map $f : D \to D$ has a fixed point, i.e. $\exists x \in D$ s.t. $f(x) = x$.

FALSE: $D \simeq \mathbb{R}^2$ $\quad D \xrightarrow[\text{homeo}]{\psi} \mathbb{R}^2$

$F: \mathbb{R}^2 \to \mathbb{R}^2$ $\quad \Rightarrow \quad \psi^{-1} \circ f \circ \psi : D \to D$

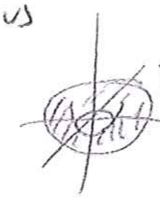
$(x,y) \quad (x+1, y)$ $\quad$ no fixed pt.

no fixed pt

→ X and Y path connected.

5b) (15 pts) If $X \cup Y$ is contractible, then X or Y is contractible.

FALSE: $X = S^1 \times D^2$ = inside of torus



$Y = \mathbb{R}^3 - X$

$\pi_1(X) = S^1$

$\pi_1(Y) = S^1$

$X \cup Y = \mathbb{R}^3$ contractible