HW #1 — SOLUTIONS

② First we prove that $v(T) - e(T) = 1$ for any tree T.
Proof by strong induction. Let the number of edges of T be n.
For $k=1$ Clearly statement is true.
Assume that for $k \leq n$ the statement is true. Then
for a tree T such that $e(T) = n+1$ delete one
of the edges. Since T is a tree this will disconnect T
(otherwise there will be a loop in T). Let T_1 and T_2 be
obtained trees. Then $e(T_1) \leq n$ and $e(T_2) \leq n$ so by
induction hypothesis $v(T_1) - e(T_1) = 1$ and $v(T_2) - e(T_2) = 1$

$v(T) = v(T_1) + v(T_2)$ and $e(T) = e(T_1) + e(T_2) + 1$

$v(T) - e(T) = v(T_1) + v(T_2) - (e(T_1) + e(T_2) + 1) = 1$

Now let Γ be a graph which is not a tree. Then there is
a spanning tree T which has the same vertices and edges
with Γ, but Γ has more edges than T since it contain
at least one loop. So result follows.

⑥ Any face has p edges also, any edge connects two faces so

$e = \frac{pf}{2}$.

Also q faces meet at each vertex implies q edges meet at
each vertex and each edge connects two vertices so

$e = \frac{qv}{2}$

Euler's formula $v - e + f = 2 \Rightarrow \frac{2e}{q} - e + \frac{2e}{p} = 2$

$\Rightarrow \frac{1}{q} + \frac{1}{p} = \frac{1}{2} + \frac{1}{e}$

(10) $f = \mathbb{R} \to (0, 1)$

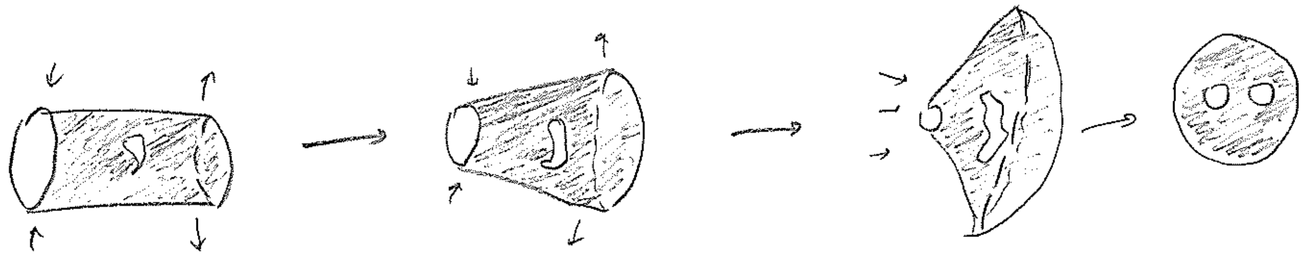
$f(x) = \frac{e^x}{1+e^x}$ clearly continous also $\lim_{x \to \infty} f(x) = 1$

and $\lim_{x \to -\infty} f(x) = 0$

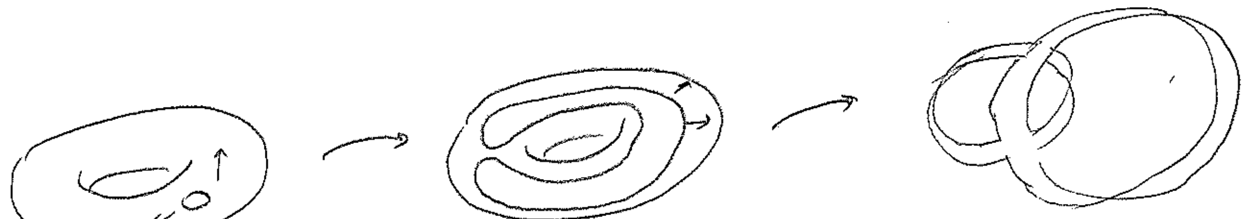
$f^{-1}(x) = \ln\left(\frac{x}{1-x}\right)$ is also continuous.

f is a homeomorphism.

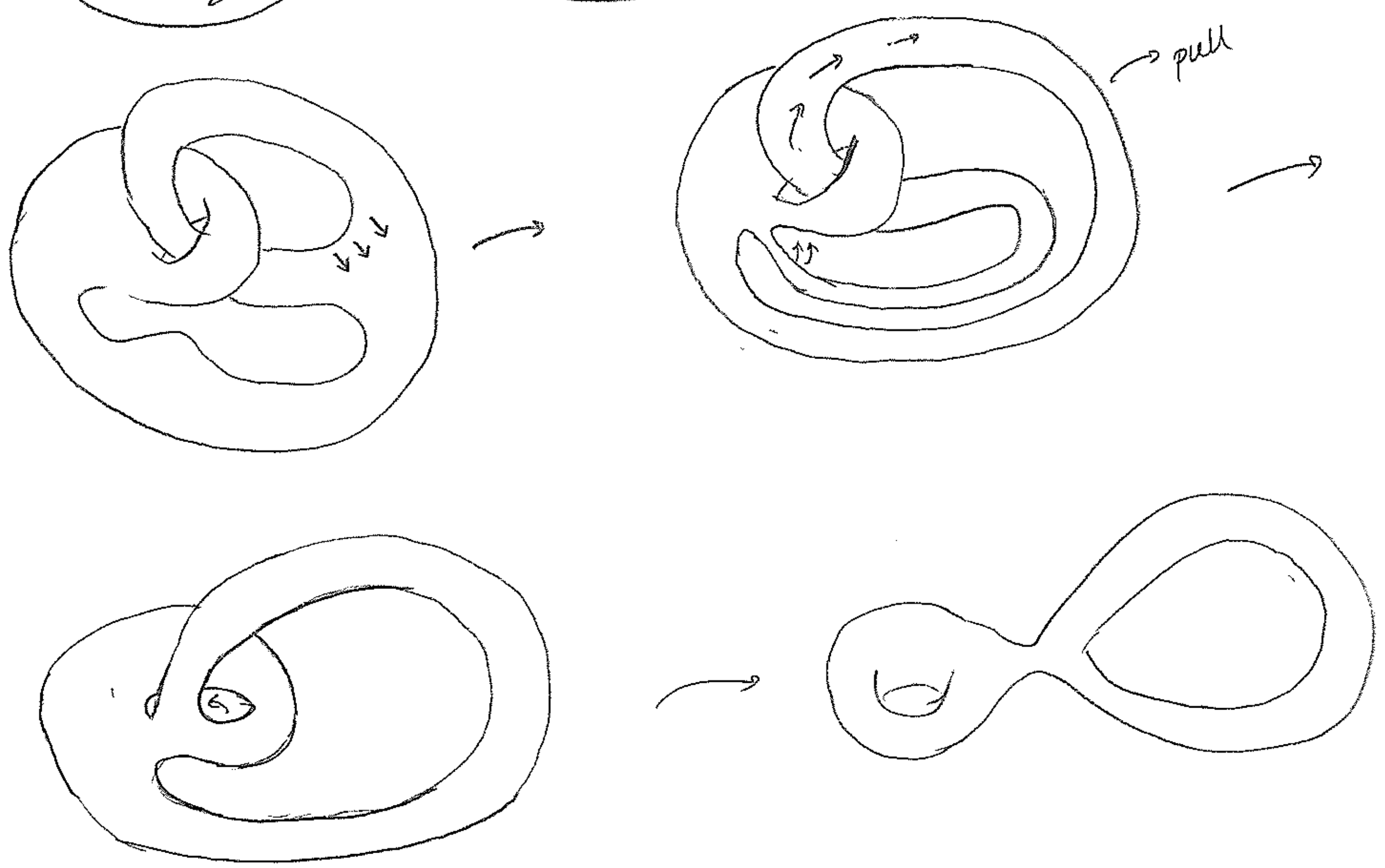
(11) (a)



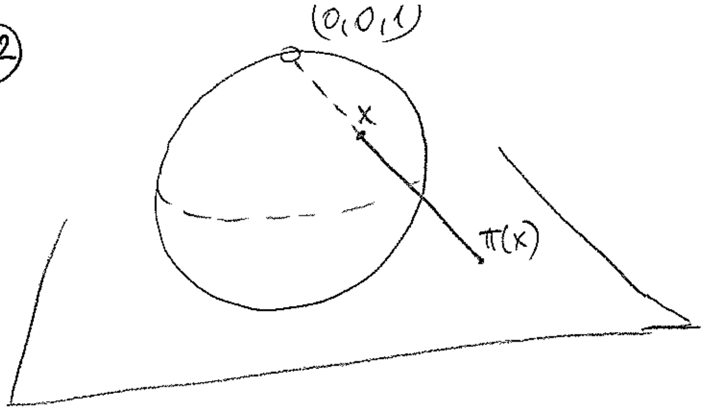
(b)



(c)



(12)



$x = (x_1, x_2, x_3)$

$\pi(x) = (a, b, 0)$

line passing through x and $\pi(x)$

$\ell(t) = x + ((x_1, x_2, x_3) - (0,0,1))\, t = (x_1 + x_1 t, x_2 + x_2 t, x_3 + (x_3 - 1)t)$

$(a, b, 0)$ is on $\ell(t)$.

$\Rightarrow x_3 + (x_3 - 1)\, t = 0 \quad \Rightarrow t = \dfrac{x_3}{1 - x_3}$

so $\pi(x) = \left(\dfrac{x_1}{1-x_3}, \dfrac{x_2}{1-x_3}, 0\right)$ which is continous since $x_3 \neq 1$.

For the inverse let $\pi(x) = (a, b, 0)$ be a point on the plane.

line passing through $(0,0,1)$ and and $\pi(x)$ is

$\ell(t) = (0,0,1) + (a, b, -1)t = (at, bt, 1-t)$

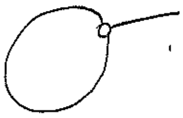
$\pi^{-1}(x) = (x_1, x_2, x_3) = (at, bt, 1-t)$

(x_1, x_2, x_3) is on the sphere so $x_1^2 + x_2^2 + x_3^2 = 1$

$(at)^2 + (bt)^2 + (1-t)^2 = 1 \quad \Rightarrow (a^2 + b^2 + 1)\, t^2 - 2t = 0 \quad t \neq 0,$

$t = \dfrac{2}{a^2 + b^2 + 1}$ so $\pi^{-1}(x) = \left(\dfrac{2a}{a^2+b^2+1}, \dfrac{2b}{a^2+b^2+1}, \dfrac{a^2+b^2-1}{a^2+b^2+1}\right)$

which is continous, so π is a homeomorphism.

(23) Assume that  and  are homeomorphic. Then there is a homeomorphism f =  $\longrightarrow$ . Consider the point  p on the circle with a spike. and $f(p)$ on the circle. Remove these points. Then the remaining sets should also be homeomophic.  and . But one is connected the other is not. ※