HW #10 – SOLUTIONS

(33) a) The 2-sphere does not have the fixed point property. Consider the map which sends every point to its antipodal. Clearly this map has no fixed point.

b)  Rotating the torus for some angle less than 360 is a map which has no fixed point.

c) The interior of the disk is homeomorphic to $\mathbb{R}^2$, say $\phi : \mathring{D} \to \mathbb{R}^2$. The translation of $\mathbb{R}^2$ has no fixed point. The map $\phi^{-1} \circ T \circ \phi$ has no fixed point.

d) One of the circles does not have fixed point property (rotation by an angle < 360). So by question 38 the one point union of two circles also doesn't have the fixed point property.

(38) Assume that both X and Y have the fixed point property. Let $f : X \vee_p Y \to X \vee_p Y$ be a map.

Define $\rho_X : X \to X \vee_p Y$, $x \mapsto x$ for $x \in X$; $\rho_Y : Y \to X \vee_p Y$, $y \mapsto y$ for $y \in Y$

and $h_X : X \vee_p Y \to X$, $x \mapsto x$ if $x \in X$, $y \mapsto p$ if $y \in Y$; $h_Y : X \vee_p Y \to Y$, $y \mapsto y$, $x \mapsto p$

Then the map $h_X \circ f \circ \rho_X : X \to X$ has a fixed point by assumption, say $x_0 \in X$. $h_X(f(\rho_X(x_0))) = x_0$

If $x_0 \neq p$: $h_X(f(\rho_X(x_0))) = h_X(f(x_0)) = x_0 \Rightarrow f(x_0) = x_0$ has fixed point.

If $x_0 = p$: $h_X(f(p)) = p \Rightarrow f(p) \in Y$.

If $f(p) = p \Rightarrow f$ has a fixed point. If $f(p) \notin X$:

Consider $h_Y \circ f \circ \rho_Y : Y \to Y$ which has a fixed point y_0. $h_Y(f(\rho_Y(y_0))) = y_0$

If $y_0 = p \Rightarrow h_Y(f(p)) = p$ not possible since $f(p) \notin X$. So $y_0 \neq p \Rightarrow$ $h_Y(f(y_0)) = y_0 \Rightarrow f(y_0) = y_0$ has fixed point.

Now for the converse assume $X \vee_p Y$ has the fixed point property. Let $f : X \to X$ be a map.
then $f_X \circ f \circ h_X : X \vee_p Y \to X \vee_p Y$ has a fixed point x_0.

$f_X(f(h_X(x_0))) = x_0 \Rightarrow f_X(f(x_0)) = x_0$ $f(x_0) \in X$ so

$\Rightarrow f(x_0) = x_0$

Y case is similar.

(40) Let $A \subseteq \mathbb{R}^n$ be compact. Then it should be bounded.
ie $\exists$ some $B_\varepsilon(x)$ s.t $A \subset B_\varepsilon(x)$. Then $\mathbb{R}^n \setminus B_\varepsilon(x) \subset \mathbb{R}^n \setminus A$.
$\mathbb{R}^n \setminus B_\varepsilon(x)$ is unbounded. $\Rightarrow$ $\mathbb{R}^n \setminus A$ is also unbounded.
which is also connected. The compact component remaining inside $B_\varepsilon(x)$ is bounded.

(49) Assume $\mathbb{E}^2$ & $\mathbb{E}^3$ are homeomorphic. Let $\phi : \mathbb{E}^3 \to \mathbb{E}^2$ be a homeomorphism. Let $x \in \mathbb{E}^3$ & $y = \phi(x) \in \mathbb{E}^2$.
Take a neighborhood $B_\varepsilon(x) \subseteq \mathbb{E}^3$. $\Rightarrow$ $\phi(B_\varepsilon(x))$ is a neigh of y.
choose δ small enough that $D_\delta(y) \subseteq \phi(B_\varepsilon(x))$.
Let $r : \mathbb{E}^2 \setminus \{y\} \to \partial D_\varepsilon(y)$ be the radial projection. Then

$r(p) = \frac{p}{\|p\|} \cdot \delta$. Then $r|_{\phi(B_\varepsilon(x)) - \{y\}} = \phi(B_\varepsilon(x) - \{y\}) \to \partial D_\delta(y)$

is a retraction. so r induce a homomorphism of $\pi_1(\phi(B_\varepsilon(x)) - \{y\})$ onto $\pi_1(\partial D_\delta(y))$.

$r_* : \pi_1(\phi(B_\varepsilon(y)) - \{y\}) \to \pi_1(\partial D_\delta(y))$

$\langle 0 \rangle \longmapsto \mathbb{Z}$

can not be onto. Contradiction. $\Rightarrow$ $\mathbb{E}^2$ and $\mathbb{E}^3$ are not homeomorphic.