HW #2 — SOLUTIONS

(8) $\mathring{A}_X = \bigcup_{\substack{O_X \in \tau_X \\ O_X \subseteq A}} O_X \qquad \mathring{A}_Y = \bigcup_{\substack{O_Y \subseteq A \\ O_Y \in \tau_Y}} O_Y$

let $z \in \mathring{A}_X = \bigcup_{\substack{O_X \in \tau_X \\ O_X \subseteq A}} O_X \Rightarrow z \in O_{X_z}$ for some $O_{X_z} \in \tau_X$ and $\subseteq A$.

$\Rightarrow O_{X_z} \cap Y \in \tau_Y$ but $O_{X_z} \cap Y = O_{X_z}$ since $O_{X_z} \subseteq A$

$\Rightarrow O_{X_z} \subseteq \mathring{A}_Y \Rightarrow z \in \mathring{A}_Y$ so $\mathring{A}_X \subseteq \mathring{A}_Y$

Consider $X = \mathbb{R}$ $\quad Y = [0,1]$ and $A = (0,1]$

$\mathring{A}_X = (0,1)$ since it is the largest open set contained in $\mathbb{R}$ with the usual topology.

But $\mathring{A}_Y = (0,1]$ since $O = (0,2)$ is an open set in $\mathbb{R}$. with the subspace topology $O \cap Y = (0,1]$ is open in Y.

So $(0,1]$ is the largest open set contained in A.

Result follows.

(12) Let $\mathcal{B}$ be a countable base for the topology of X

$\mathcal{B} = \{O_1, O_2, \ldots\}$ and every open set in X is a union of members of $\mathcal{B}$, since it is a base.

set $A = \{x_n : x_n \in O_n\}$ is clearly countable since it contains at most one element from each open set.

Claim: A is dense in X.

Let $O \subseteq X$ be an open set in X containing some arbitrary $x \in X$. which means that so O is a union of some members of $\mathcal{B}$

the intersection of A with O is nonempty. Result follows.

(15) Let $f: \mathbb{E}^1 \to \mathbb{E}^1$ be a map.
and $\Gamma_f = \mathbb{E}^1 \to \mathbb{E}^2$ by $\Gamma_f(x) = (x, f(x))$
Let $O \subseteq \mathbb{E}^2$ be an open set. Then by the product
topology it can be written as a union of open sets
since all these sets $U \times V$ form a basis for $\mathbb{E}^2$

$\bigcup (U \times V)$

U open in $\mathbb{E}^1$
V open in $\mathbb{E}^1$

We need to show that $\Gamma_f^{-1}(O)$ is open in $\mathbb{E}^1$. $\Gamma_f^{-1}(O) =$

$$\Gamma_f^{-1}\Big(\bigcup_{\substack{U \subseteq \tau(\mathbb{E}^1) \\ V \subseteq \tau(\mathbb{E}^1)}} U \times V\Big) = \bigcup_{\substack{U \subseteq \tau(\mathbb{E}^1) \\ V \subseteq \tau(\mathbb{E}^1)}} \Gamma_f^{-1}(U \times V) = \bigcup_{U \subseteq \tau(\mathbb{E}^1)} U \quad \text{open}$$

$\forall\, U, V \in \tau(\mathbb{E}^1)$

and $V = f(U)$. $U = f^{-1}(V)$

(and V open $\Rightarrow$ U open true when f is cts)

so $\Gamma_f^{-1}(O)$ is open. $\Rightarrow$ Γ_f is continuous. Now we need
to show that Γ_f is a homeomorphism onto its image.

$\Gamma_f = \mathbb{E}_1 \to \Gamma_f(\mathbb{E}_1)$

Let $O \subset \mathbb{E}_1$ be an open set. Then it can be written as
a union of open intervals (a,b). since they constitute a basis
for $\mathbb{E}^1$. $O = \bigcup (a,b)$

$\Gamma_f(O) = \Gamma_f(\bigcup (a,b)) = \bigcup \Gamma_f(a,b)$ by subspace topology $\Gamma_f(a,b)$

$$\Gamma_f(a,b) = \underbrace{(a,b) \times \mathbb{E}^1}_{\text{open in } \mathbb{R}^2} \cap \Gamma_f(\mathbb{E}^1)$$

is open in $\Gamma_f(\mathbb{E}^1)$. so $\Gamma_f(O)$ is also open being arbitrary
union of open sets.
Γ_f is an open map. Γ_f trivially 1-1 and also onto its image.

(27) Show $d(x,A) = 0$ iff x is a point of $\bar{A}$.

Assume that x is a point of $\bar{A}$. Assume for a contradiction that $d(x,A) = \varepsilon > 0$

since $x \in \bar{A}$ $\quad B_{\frac{\varepsilon}{2}}(x) \cap A \neq \phi \quad \exists a_\varepsilon \in B_{\frac{\varepsilon}{2}}(x) \cap A$

$\Rightarrow d(x, a_\varepsilon) < \frac{\varepsilon}{2}$ but $\inf_{a \in A} d(x,a) = \varepsilon \geq \frac{\varepsilon}{2}$ contradiction.

Conversely assume that $d(x, A) = 0$ and $\varepsilon > 0$ be given

$\inf_{a \in A} d(x,a) = 0 \Rightarrow \inf_{x \in A} d(x,a) + \varepsilon > d(x, a_\varepsilon)$ for some

$a_\varepsilon \in A \Rightarrow \varepsilon > d(x, a_\varepsilon) \Rightarrow a_\varepsilon \in B_\varepsilon(x)$

$\Rightarrow x \in \bar{A}$

(22) http://en.wikipedia.org/wiki/Space_filling_curve