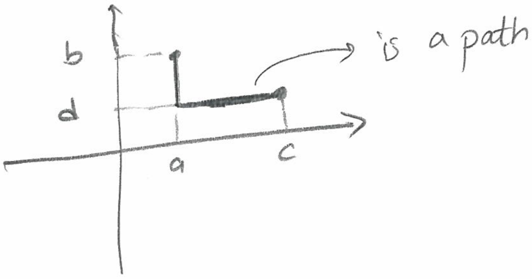
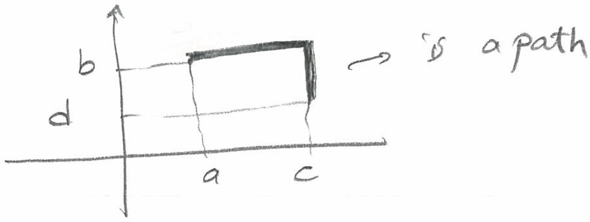
HW #4 – SOLUTIONS

30 Let (a,b) and (c,d) be two points in X.

For $a \in \mathbb{Q}$ $d \in \mathbb{Q}$



For $b \in \mathbb{Q}$ $c \in \mathbb{Q}$



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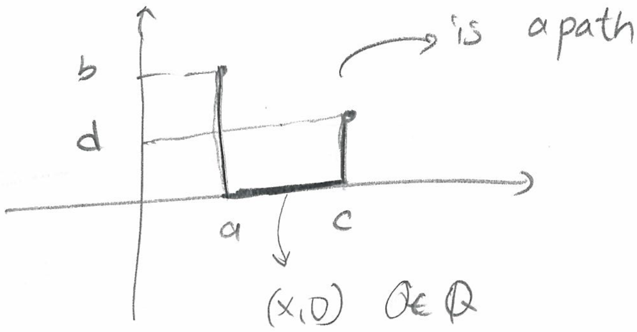

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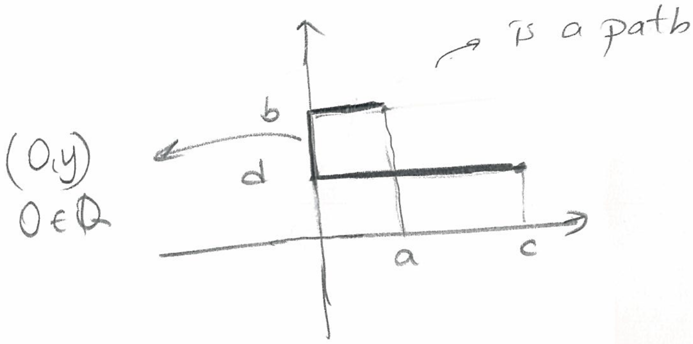


path connected $\Rightarrow$ connected

34 Claim 1: Any Euclidean space E^n is locally connected.

Let $x \in E^n$ be a point and N be a neighborhood of x.

$x = (x_1, x_2, \dots, x_n)$ Also consider the projection maps

$p_i(x) = x_i \quad \forall\, 1 \le i \le n$

$p_i(N) = N_i$ is a neighborhood of x_i in $\mathbb{R}$. $\forall i$.

so $\exists\, \varepsilon_i$ such that $B_{\varepsilon_i} \subseteq N_i$ then we know $B_\varepsilon(x_i) \subseteq \mathbb{R}$ is connecte

Consider $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_n\}$

$B_\varepsilon(x_1) \times B_\varepsilon(x_2) \times \dots \times B_\varepsilon(x_n)$ is also connected by theorem

3.26.

Claim 2: Any locally Euclidean space M is locally connected.

Let $x \in M$ be a point and $x \in N$ be a neighborhood of it. We know that there is a neighborhood N_x of x which is homeomorphic to $\mathbb{E}^n$. WLOG assume $N_x \subseteq N$.

$\exists\, U \subseteq \mathbb{E}^n$ and a continuous map $h: U \to N_x$ is a homeomorphism. $h^{-1}(x) = y \in U$ so $\exists$ a neighborhood U_y of y which is connected and $U_y \subseteq U$ so $h(U_y) \subseteq N_x$ is also connected since h is continuous.

Claim 3: $X = \{0\} \cup \{1/n \mid n = 1, 2, \dots\}$ is not locally connected.

Consider $0 \in X$. Let N be a neighborhood of 0. Assume that $\exists\, \varepsilon > 0$ $B_\varepsilon(0) \ni 0$ is connected. If $\frac{1}{N} < \varepsilon$ such that N is minimum among the numbers satisfying this then

Consider $\varepsilon_N < \frac{1}{N+1} - \frac{1}{N}$

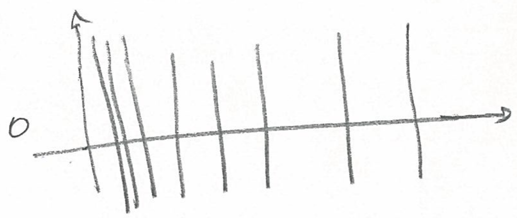
For this ε_N $B_{\varepsilon_N}(\frac{1}{N})$ is an open ball which is connected

$B_{\varepsilon - \varepsilon_N}(0)$ is open and $B_{\varepsilon - \varepsilon_N}(0) \cap B_{\varepsilon_N}(\frac{1}{N}) = \emptyset$

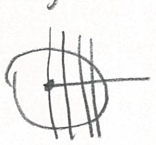
and their union is $B_\varepsilon(0)$ so it is not connected.

43) $X = \bigcup \{\frac{1}{n} : n = 1, 2, \dots\}$

Consider the set $\bigcup \frac{1}{n} \times \mathbb{R}$



This is path connected because when we take two points from the set we can go from one to another following a path. Now consider $(0,0)$ and a neighborhood N of $(0,0)$ for some $\varepsilon > 0$, $B_\varepsilon(0,0) \subseteq N$.



But similar to the previous argument we can write it as a union of two disjoint open sets so it is not connected $\Rightarrow$ not path connected

(44) Let X be a space which is connected and locally path connected. Let $x \in X$ be a point.

Describe a set

$U(x) = \{ w \in X$: there exist a path from x to $w \}$

Let $w_0 \in U(x)$ since it is locally path connected

$\exists$ a neighborhood N_{w_0} of w_0 such that which is path connected but $N_{w_0} \subseteq U(x)$ since $\forall w \in N_{w_0}$ there is a path from w_0 to w and also since $w_0 \in U(x)$ there is a path from x to w_0 connect these paths we can get a path from x to w.

$\Rightarrow N_{w_0} \subseteq U(x)$ follows which means that $U(x)$ is open.

Let $y \in U(x)^c$ so there is no path from x to y.

But X is locally path connected so there is a path connected neighborhood N_y of y.

$N_y \subseteq U(x)^c$ otherwise $\nexists$ for $t \in N_y$ there is a path from x to t means there is a path from x to y $\Rightarrow y \in U(x)$ which is a contradiction.

So $U(x)$ is both closed and open

since X is connected the only choice is

$$U(x) = X$$

$\therefore X$ is path connected.