Math 402/571 Topology

Midterm 1

March 24, 2010

1) (5 pts each) For each of (a)-(d) below: If the proposition is true, write TRUE. If the proposition is false, write FALSE. No explanations are required for this problem.

1a) Let $f : X \to Y$ be continuous. If A is compact in Y, then $f^{-1}(A)$ is compact in X.

1b) Let $f : X \to Y$ be a continuous bijection. If X is Hausdorff and Y is compact, then f is a homeomorphism.

1c) If $X \times Y$ is homeomorphic to $X \times Z$, then Y is homeomorphic to Z.

1d) Let (X, τ) be a topological space, and let $A \subset X$ be both open and closed in X. Then, A is a component of X.

2) (20 pts) Show that every metric space is normal.
i.e. Let (X, d) be a metric space. If A, B are two disjoint closed subsets of X, then there are disjoint open sets O_A, O_B with $A \subset O_A$ and $B \subset O_B$.

3) (20 pts) Give an example of two different topologies τ_1 and τ_2 on the same set X such that identity map I is not continuous in either direction.
i.e. $I_1 : (X, \tau_1) \to (X, \tau_2)$ and $I_2 : (X, \tau_2) \to (X, \tau_1)$ are not continuous.

4) Prove or give a counterexample for the following statements:
Let (X, d) be a metric space, and $A \subset X$.

4a) (10 pts) If A is compact, then A is closed and bounded.

4b) (10 pts) If A is closed and bounded, then A is compact.

5) Prove or give a counterexample for the following statements:
Let (X, τ) be a topological space, and $A \subset X$.

5a) (10 pts) If A is path connected, then $\overline{A}$ is path connected.

5b) (10 pts) If A is connected, then $\overline{A}$ is connected.

KEY

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1) (5 pts each) For each of (a)-(d) below: If the proposition is true, write TRUE. If the proposition is false, write FALSE. No explanations are required for this problem.

1a) Let $f : X \to Y$ be continuous. If A is compact in Y, then $f^{-1}(A)$ is compact in X.

False. $f: \mathbb{R} \to \{p\}$

1b) Let $f : X \to Y$ be a continuous bijection. If X is Hausdorff and Y is compact, then f is a homeomorphism.

False. X compact, Y Hausdorff $\Rightarrow$ ✓

Ex: $f: [0, 2\pi) \to S^1$, $\theta \mapsto (\cos\theta, \sin\theta)$. f cts, bij. no homeo.

1c) If $X \times Y$ is homeomorphic to $X \times Z$, then Y is homeomorphic to Z.

False. $X = [0,1]$, $Y = [0,1)$, $Z = [0,1]$. $X \times Y \simeq X \times Z$ but $Y \not\simeq Z$

1d) Let (X, τ) be a topological space, and let $A \subset X$ be both open and closed in X. Then, A is a component of X.

False. Need A connected, too.

Ex: $X = \{0, 1, 2\}$

$A = \{0, 1\}$ both open and closed. but not a component!

2) (20 pts) Show that every metric space is normal.
i.e. Let (X, d) be a metric space. If A, B are two disjoint closed subsets of X, then there are disjoint open sets O_A, O_B with $A \subset O_A$ and $B \subset O_B$.

Let $O_A = \{ x \in X \mid d(x,A) < d(x,B) \}$

$O_B = \{ x \in X \mid d(x,B) < d(x,A) \}$

Then $A \subseteq O_A$. Since $\boxed{d(x,A) = 0 \Leftrightarrow x \in \bar{A}}$

$x \in A \Rightarrow d(x,A) = 0 < d(x,B) \Rightarrow x \in O_A$

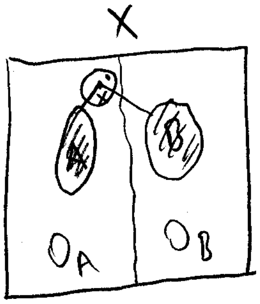
similarly $B \subseteq O_B$

Claim: O_A is open.

Since (X,d) metric space it is enough to show that

$\forall x \in O_A \ \exists \varepsilon_x > 0 \ B_{\varepsilon_x}(x) \subseteq O_A$.

Let $x \in O_A$. Let $\delta_x = d(x,B) - d(x,A) > 0$



let $\varepsilon_x = \frac{\delta_x}{3}$

Consider $B_{\varepsilon_x}(x)$. let $y \in B_{\varepsilon_x}(x)$

$d(y,A) \leq d(x,y) + d(x,A)$ (show)

$d(x,B) \leq d(x,y) + d(y,B)$

$\Rightarrow d(y,A) \leq d(x,y) + d(x,A) = d(x,y) + d(x,B) - \delta_x \leq \underbrace{2d(x,y) - \delta_x}_{<0} + d(y,B)$

$\Rightarrow d(y,A) < d(y,B) \Rightarrow y \in O_A \Rightarrow B_{\varepsilon_x}(x) \subseteq O_A \Rightarrow O_A$ open.

similarly O_B open. By definition, $O_A \cap O_B = \emptyset$ $\square$

3) (20 pts) Give an example of two different topologies τ_1 and τ_2 on the same set X such that identity map I is not continuous in either direction. i.e. $I_1 : (X, \tau_1) \to (X, \tau_2)$ and $I_2 : (X, \tau_2) \to (X, \tau_1)$ are not continuous.

$X = \{p, q\}$

$T_1 = \{\emptyset, X, \{p\}\}$

$T_2 = \{\emptyset, X, \{q\}\}$

$id : (X, T_1) \to (X, T_2)$ not cts since

$\{q\}$ open in (X, T_2) $id^{-1}(\{q\}) = \{q\}$ is not open in (X, T_1)

$id : (X, T_2) \to (X, T_1)$ not cts, since

$\{p\}$ open in (X, T_1). $id^{-1}(\{p\}) = \{p\}$ is not open in (X, T_2).

$\Box$.

4) Prove or give a counterexample for the following statements:

Let (X, d) be a metric space, and $A \subset X$.

4a) (10 pts) If A is compact, then A is closed and bounded.

Proof: X metric space $\Rightarrow$ X Hausdorff.

X Hausdorff + A compact $\Rightarrow$ A closed.

A bounded since $F = \{B_n(p)\}$ open covering for A.

There is a finite subcover $\Rightarrow$ $\exists N$ $A \subseteq B_N(p)$

4b) (10 pts) If A is closed and bounded, then A is compact.

No. $X = [0,1]$ d discrete metric $\left(\forall x, y \atop x \neq y \;\; d(x,y) = 1\right)$

$A = X \Rightarrow$ A is closed ✓

A is bounded since $A \subseteq B_2(0)$.

A is not compact since

$F = \{\{p\} \mid p \in [0,1]\}$ is an open covering

with no finite subcover!

5) Prove or give a counterexample for the following statements:

Let (X, τ) be a topological space, and $A \subset X$.

5a) (10 pts) If A is path connected, then $\overline{A}$ is path connected.

No. A topologist sine curve in $\mathbb{R}^2$.

$A = \{ (\theta, \sin\theta) \mid \theta \in (0,1] \}$

A path connected ✓

$\bar{A} = A \cup B$ where $D = \{ (0,t) \mid t \in [-1,1] \}$

$A \cup D$ is not path connected! (class notes)

5b) (10 pts) If A is connected, then $\overline{A}$ is connected.

Yes. Assume $\bar{A}$ is not connected.

Then $\exists Y \subset \bar{A}$, $Y \neq \emptyset$, $Y \neq \bar{A}$ both open and closed in $\bar{A}$.

Consider $Y \cap A$. $Y \cap A \neq \emptyset$ since A is dense in $\bar{A}$ and Y is open in $\bar{A}$. $\Rightarrow Y \cap A \neq \emptyset$

$Y \cap A \neq A$ since Y^c is open in $\bar{A}$ and A is dense in A $\Rightarrow Y^c \cap A \neq \emptyset$ $\Rightarrow A \not\subset Y$.

$Y \cap A$ is both open and closed in A by subspace topology

$\times$ since A is connected.