Math 402/571 Topology

Midterm 2

May 26, 2010

1a) (5 pts) Define the fundamental group, $\pi_1(X, p)$.
1b) (5 pts) Define homotopy between two maps. Define homotopy equivalence of two spaces.
1c) (5 pts) Define identification space. Define identification topology.
1d) (5 pts) Define surface. Define boundary point and interior point of a surface.

2) (5 pts each) For each of (a)-(d) below: If the proposition is true, write TRUE. If the proposition is false, write FALSE. No explanations are required for this problem.
2a) If $\pi_1(X, p) \simeq \pi_1(X, q)$ for any $p, q \in X$, then X is path connected.
2b) Let A be a subspace of the topological space X. If $f : X \to A$ is a retraction, then $f_* : \pi(X, p) \to \pi_1(A, p)$ is onto where $p \in A$.
2c) Let Y be a contractible space. Then any two maps $f, g : X \to Y$ are homotopic to each other.
2d) $X - \{p\} \simeq Y - \{q\} \quad \Rightarrow \quad X \sim Y$.
i.e. Let $p \in X$ and $q \in Y$. If $X - \{p\}$ is homeomorphic to $Y - \{q\}$, then X is homotopy equivalent to Y.

3) (20 pts) Show that $int(D^2)$ is not homeomorphic to $int(D^3)$.
$int(D^2) = \{(x, y) \in \mathbf{R}^2 \mid |(x, y)| < 1 \}$
$int(D^3) = \{(x, y, z) \in \mathbf{R}^3 \mid |(x, y, z)| < 1 \}$

4) (20 pts) Show that $\pi_1(S^1) = \mathbf{Z}$.
You may use any method you want.

5) (20 pts) Prove or give a counterexample for the following statement:
If $\pi_1(X) \simeq \mathbf{Z}$, then X is homotopy equivalent to S^1.

Bonus) (20 pts) Let $X = T^2 - \{p, q\}$ where p and q are two distinct points on the torus T^2. Compute $\pi_1(X)$.

2) (5 pts each) For each of (a)-(d) below: If the proposition is true, write TRUE. If the proposition is false, write FALSE. No explanations are required for this problem.

2a) If $\pi_1(X,p) \simeq \pi_1(X,q)$ for any $p,q \in X$, then X is path connected.

FALSE.

Ex: $X = \{p, q\}$

2b) Let A be a subspace of the topological space X. If $f : X \longrightarrow A$ is a retraction, then $f_* : \pi(X,p) \longrightarrow \pi_1(A,p)$ is onto where $p \in A$.

TRUE

2c) Let Y be a contractible space. Then any two maps $f, g : X \longrightarrow Y$ are homotopic to each other.

TRUE

2d) $X - \{p\} \simeq Y - \{q\} \quad \Rightarrow \quad X \sim Y$.

i.e. Let $p \in X$ and $q \in Y$. If $X - \{p\}$ is homeomorphic to $Y - \{q\}$, then X is homotopy equivalent to Y.

FALSE.

$X = S^1$

$Y = [0,1) \quad q = \{0\}$

3) (20 pts) Show that $int(D^2)$ is not homeomorphic to $int(D^3)$.

$$int(D^2) = \{(x,y) \in \mathbf{R}^2 \mid |(x,y)| < 1\}$$
$$int(D^3) = \{(x,y,z) \in \mathbf{R}^3 \mid |(x,y,z)| < 1\}$$

Assume $\mathring{D}^2 \cong \mathring{D}^3$

Let $\varphi : \mathring{D}^2 \to \mathring{D}^3$ homeomorphism.

Let $\varphi(0) = q \in \mathring{D}^3$

Then $\varphi|_{\mathring{D}^2 - \{0\}} : \mathring{D}^2 - \{0\} \to \mathring{D}^3 - \{q\}$ homeo.

However, $\mathring{D}^2 - \{0\} \sim S^1$ check

$\mathring{D}^3 - \{q\} \sim S^2$ check

$\Rightarrow$ $\pi_1(\mathring{D}^2 - \{0\}) = \mathbb{Z}$

$\pi_1(\mathring{D}^3 - \{q\}) = \langle 0 \rangle$

Since $\mathbb{Z} \not\cong \langle 0 \rangle \Rightarrow \mathring{D}^2 - \{0\} \not\cong \mathring{D}^3 - \{q\} \Rightarrow \mathring{D}^2 \not\cong \mathring{D}^3$

4) (20 pts) Show that $\pi_1(S^1) = \mathbf{Z}$.

You may use any method you want.

Claim: $\mathbb{R}/\mathbb{Z} \simeq S^1$

$$f : \mathbb{R} \to S^1$$
$$\theta \quad e^{2\pi i\theta}$$

f is identification map, since f sends open sets to open sets.

$\forall s \in S^1$ $f^{-1}(s) = \{t+n \mid n \in \mathbb{Z}\}$ for some t with $s = e^{2\pi i t}$.

$\Rightarrow$ $P = \{\{f^{-1}(s)\} \mid s \in S^1\}$ is the same partition with the orbit space of $\mathbb{R}/\mathbb{Z}$.

$\Rightarrow$ $\mathbb{R}/\mathbb{Z} \simeq S^1$

$\mathbb{R}$ simply conn. $\mathbb{Z} \curvearrowright \mathbb{R}$ satisfies the conditions

$\Rightarrow$ $\pi_1(S^1) = \pi_1(\mathbb{R}/\mathbb{Z}) = \mathbb{Z}$.

[Theorem 5.13]

5) (20 pts) Prove or give a counterexample for the following statement:

If $\pi_1(X) \simeq \mathbf{Z}$, then X is homotopy equivalent to S^1.

FALSE.

$Y = S^2 \vee S^1$

$\pi_1(Y) = \mathbb{Z}$ ✓

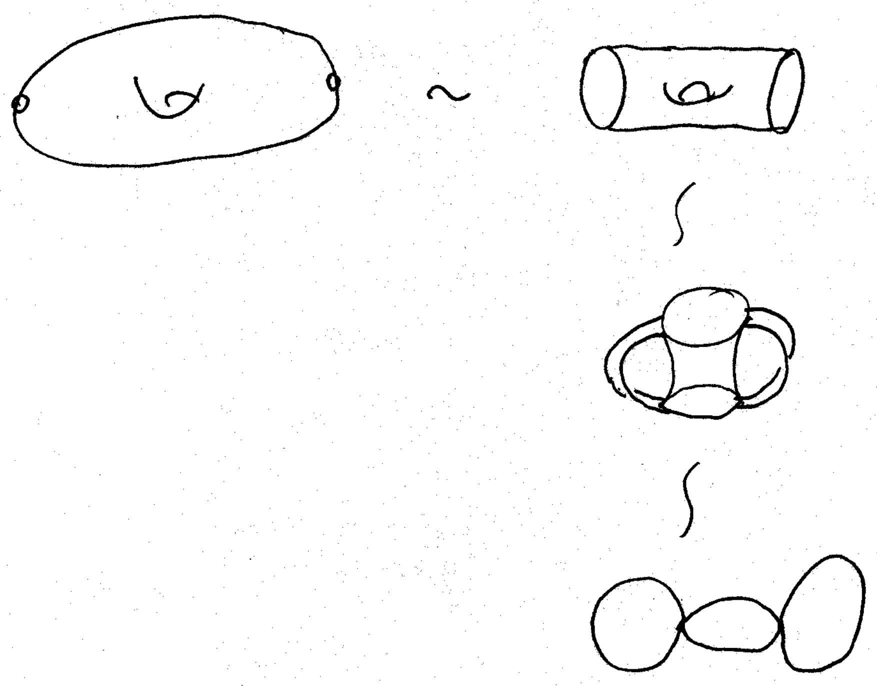
$\chi(Y) = 1$

$\Rightarrow \quad Y \not\simeq S^1$

$\chi(S^1) = 0$

Euler Characteristics.

Bonus) (20 pts) Let $X = T^2 - \{p, q\}$ where p and q are two distinct points on the torus T^2. Compute $\pi_1(X)$.



$S^1 \vee S^1 \vee S^1$

$\pi_1(T^2 - \{p,q\}) = \pi_1(S^1 \vee S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$