KEY

Math 405/538 Differential Geometry
Midterm Exam 1

November 6, 2013

1) (5 pts each)

a. Define the torsion of a regular space curve α.

textbook.

b. Define the tangent plane of a regular surface in $\mathbb{R}^3$.

textbook

For each of (c)-(d) below: If the proposition is true, write TRUE. If the proposition is false, write FALSE. No explanations are required.

c. There exists exactly one regular parametrized curve $\alpha : \mathbb{R} \to \mathbb{R}^3$ satisfying all of the following conditions:

- α is parametrized by arc length.
- $\alpha(0) = (0, 0, 0)$
- α has curvature $\kappa(s) = 5s$ for all $s \in \mathbb{R}$
- α has torsion $\tau(s) = 2$ for all $s \in \mathbb{R}$

FALSE. $\alpha'(0)$ not specified. φ rigid motion $\Rightarrow$ α ✓, $\varphi(\alpha)$ ✓

d. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a differentiable function. If a is not a regular value of f, then $f^{-1}(a)$ is not a surface in $\mathbb{R}^3$.

FALSE: $f(x,y,z) = x^2$
0 not regular but $f^{-1}(0) = \{x=0\}$ is surface.

2) (20 pts) Prove or give a counterexample for the following statement:

A regular space curve is planar if and only if its torsion τ is everywhere 0.

AMBIGUITY: ① (Lecture Notes: [α reg. $\Leftrightarrow$ $\alpha' \neq 0, \alpha'' \neq 0$] ($\exists$ Frenet frame)
② (Textbook: [α reg $\Leftrightarrow$ $\alpha' \neq 0$] (No frene frenet frame for $\alpha'' = 0$)
$k = 0$

For ① ($\Rightarrow$)) $\alpha(t) \subseteq P_0$. Let N_0 normal to P_0. $\Rightarrow$ $\alpha(t) \cdot N_0 = C$

$\Rightarrow$ $\alpha'(t) \cdot N_0 = 0$, $\alpha''(t) \cdot N_0 = 0$ $\Rightarrow$ $\alpha' \times \alpha'' = B(t) = N_0$ $\Rightarrow$ $B'(t) = 0$ $\Rightarrow$ $\tau = 0$

① ($\Leftarrow$) $\tau = 0 \Rightarrow B' = 0 \Rightarrow B(t) = B_0 \Rightarrow \alpha'(t) \cdot B_0 = 0 \Rightarrow \alpha(t) \cdot B_0 = C$

$\alpha(t) \cdot B_0 = C \Rightarrow \alpha(t) \subseteq P_0$ for some plane P_0 with normal B_0.
(i.e. $\langle \alpha(t) - q_0, B_0 \rangle = 0$ for some q_0 with $B_0 \cdot q_0 = C$

For ② ($\Rightarrow$)) ✓ Same as above.

($\Leftarrow$) Counterexample: P24 - ex. 10
($\alpha'' = 0$ at some point. (not regular w.r.t. ①))

3) (20 pts) Assume that all normals of a space curve α pass through the fixed point $p_0 \in \mathbf{R}^3$. Then, show that α is contained in a circle.

$$\alpha(t) + \lambda(t)N(t) = p_0$$

$$\alpha'(t) + \lambda' N + \lambda N' = 0$$

$$T + \lambda' N + \lambda(-kT - \tau B) = 0 \Rightarrow (1-\lambda k)T + \lambda' N - \lambda\tau B = 0$$

$\{T, N, B\}$ Orthonormal $\Rightarrow$ $1-\lambda k = 0$, $\lambda' = 0$, $\lambda\tau = 0$ $\Rightarrow$ $\lambda = 1/k$, λ = constant, $\tau = 0$ $\Rightarrow$ $k = \frac{1}{\lambda_0}$

$[\tau = 0 \Rightarrow \alpha$ planar$]$ α planar + k constant $\Rightarrow$ α is a circle of radius $\frac{1}{k} = \lambda_0$ by fondamental Theorem of Space Curves

4) (20 pts) Let $S = \{x^2 + 4y^2 + z^3 = 5\}$. Show that S is a regular surface in $\mathbb{R}^3$. Find the tangent plane at $p = (1, 1, 0)$. Write a parametrization of a neighborhood of p.

S regular: $f(x,y,z) = x^2 + 4y^2 + z^3 \Rightarrow \nabla f = \langle 2x, 8y, 3z^2 \rangle$

$\nabla f = (0,0,0) \Rightarrow \begin{matrix} x=0 \\ y=0 \\ z=0 \end{matrix} \Rightarrow f(0,0,0) = 0$ critical value

$\mathbb{R} - \{0\}$ regular value.

$\Rightarrow f^{-1}(5) = S$ regular surface.

$\nabla f(1,1,0) = \langle 2, 8, 0 \rangle =$ normal

$2(x-1) + 8(y-1) + 0(z-0) = 0 \Rightarrow \boxed{2x + 8y = 10}$

Tangent plane.

$x = \sqrt{5 - 4y^2 - z^3} \Rightarrow$ let $U \subseteq yz$-plane nbhd of $\underset{y,z}{(1,0)}$

$\varphi : U \longrightarrow \mathbb{R}^3$ parametrization of a nbd

$(y,z) \quad (\sqrt{5 - 4y^2 - z^3}, y, z)$ of $p = (1,1,0)$ in S.

Note: $\varphi(x,y) = (x, y, \sqrt{5 - x^2 - 4y^2})$ is not a parametrization near $(1,1)$ since φ is not diff. at $(1,1)$.

5) (20 pts) Let S be a regular surface and $\varphi : U \to S$ be a parametrization with $\varphi(p) = q$. Let the coefficients of the first fundamental form for φ at q are $E = 3$, $F = -2$ and $G = 7$. Let α_1, α_2 be two regular curves in $U \subset \mathbb{R}^2$ with $\alpha_1(0) = \alpha_2(0) = p$ and $\alpha_1'(0) =< 1, 0 >$ and $\alpha_2'(0) =< 1, 1 >$.

a) Compute $\cos\theta$, where θ is the angle between the images of these two curves in S at q.

Let $v = \langle a, b \rangle \in \mathbb{R}^2 \Rightarrow |v|^2 = I_q(a,b) = 3a^2 - 4ab + 7b^2$

$\Rightarrow v_1 = \alpha_1'(0) = \langle 1, 0 \rangle \Rightarrow |v_1|^2 = 3$

$v_2 = \alpha_2'(0) = \langle 1, 1 \rangle \Rightarrow |v_2|^2 = 3 - 4 + 7 = 6$

$v_1 \cdot v_2 = [1 \; 0] \begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = [1 \; 0] \begin{bmatrix} 1 \\ 5 \end{bmatrix} = 1$

$\cos\theta = \dfrac{v_1 \cdot v_2}{|v_1||v_2|} = \dfrac{1}{\sqrt{3} \cdot \sqrt{6}} = \dfrac{1}{3\sqrt{2}}$ ✓

b) Let $\beta(t) = (t^2 + 1, 2t - 3) \subset U$ and $\beta(0) = p$. Let $\gamma(t) = \varphi(\beta(t)) \subset S$. Compute $|\gamma'(0)|$.

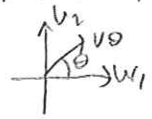
$\beta'(t) = \langle 2t, 2 \rangle \Rightarrow \beta'(0) = \langle 0, 2 \rangle \Rightarrow \gamma'(0) = 0\varphi_u + 2\varphi_v$

$\Rightarrow |\gamma'(0)|^2 = 0 - 0 + 7 \cdot 4 = 28$

$|\gamma'(0)| = 2\sqrt{7}$ ✓

Bonus) a) (10 pts) Show that the sum of the curvatures of the normal sections ($\pm k_\theta$) for any pair of orthogonal directions is constant.

$\pm k_\theta = k_1 \cos^2\theta + k_2 \sin^2\theta$ (k_1, k_2 principal curvatures, w_1, w_2 principal directions)



consider α and $\alpha \pm \pi/2$

$\pm k_\alpha = k_1 \cos^2\alpha + k_2 \sin^2\alpha$

$+ \quad \pm k_{\alpha+\pi/2} : k_1 \cos^2(\alpha + \pi/2) + k_2 \sin^2(\alpha + \pi/2) = k_1 \sin^2\alpha + k_2 \cos^2\alpha$

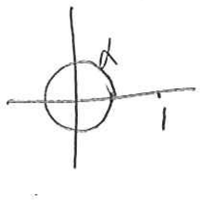
$\Rightarrow \quad = k_1(\cos^2\alpha + \sin^2\alpha) + k_2(\cos^2\alpha + \sin^2\alpha) = k_1 + k_2 \quad \square$

b) (10 pts) Prove or give a counterexample for the following statement:

Let S be a regular surface with the principal curvatures $|k_1| \leq 1$ and $|k_2| \leq 1$ everywhere. Then, for any curve $\alpha \subset S$, the curvature of α satisfies $|k_\alpha| \leq 1$.

Counterexample: $P = xy$-plane

$\alpha(t) = \left(\frac{1}{2}\cos t, \frac{1}{2}\sin t, 0\right)$



$k_\alpha = 2$