Math 405/538 Differential Geometry
Midterm Exam 2

December 18, 2013

1a) (5 pts) Define covariant derivative.
1b) (5 pts) Define geodesic.
1c) (5 pts) State Gauss' Theorem (Theorem Egregium).
1d) (5 pts) Define holonomy.

2) (5 pts each) For each of (a)-(d) below: If the proposition is true, write TRUE. If the proposition is false, write FALSE. No explanations are required for this problem.

2a) Let L be a straight line in a surface S in $\mathbb{R}^3$. Then, L is a geodesic of S.
2b) Let S_1 and S_2 be two isometric surfaces. Then the mean curvatures for corresponding points are same.
2c) Let S be a surface in $\mathbb{R}^3$, which is homeomorphic to a torus. Then, there are points in S such that the Gaussian curvature is positive, negative and zero.
2d) Let S_1 and S_2 be isometric surfaces in $\mathbf{R}^3$ where the second fundamental forms are same. Then there is a rigid motion mapping S_1 onto S_2.

3) (20 pts) Construct a surface S which contains a point p such that the principal curvatures at p are 4 and -3.
Compute the first and second fundamental form at p for S.
Verify the Gaussian curvature at p.

4) (20 pts) Let S be a round sphere with radius 2 in $\mathbb{R}^3$, and let $p \in S$.
Show that there exists a curve $\alpha_k \subset S$ through p where its curvature at p is equal to k if and only if $k \geq \frac{1}{2}$.

5a) (10 pts) Let α be regular curve in a surface S. Show that α is a line of curvature if and only if $N'(t) = \lambda(t)\alpha'(t)$ where $N(t) = N(\alpha(t))$ and $\lambda(t)$ is a differentiable function.
5b) (10 pts) Show that if $\alpha \subset S$ is both a line of curvature and a geodesic, then α is a plane curve.

6a) (13 pts) Show that the surfaces

$$\varphi(u,v) = (u\cos v, u\sin v, \log u) \qquad \psi(u,v) = (u\cos v, u\sin v, v)$$

have equal Gaussian curvature at the points $\varphi(u,v)$ and $\psi(u,v)$.
Hint: $K = \frac{eg-f^2}{EG-F^2}$
6b) (7 pts) Give a counterexample to the converse of the Gauss Theorem.
Hint: Show $\varphi \circ \psi^{-1}$ is not an isometry.

2) (5 pts each) For each of (a)-(d) below: If the proposition is true, write TRUE. If the proposition is false, write FALSE. No explanations are required for this problem.

2a) Let L be a straight line in a surface S in $\mathbb{R}^3$. Then, L is a geodesic of S.

TRUE. $k = 0 \Rightarrow k_g = 0$

2b) Let S_1 and S_2 be two isometric surfaces. Then the mean curvatures for corresponding points are same.

FALSE: Plane vs. Cylinder

2c) Let S be a surface in $\mathbb{R}^3$, which is homeomorphic to a torus. Then, there are points in S such that the Gaussian curvature is positive, negative and zero.

TRUE: $\iint K \, dA = 2\pi \chi(S) = 0$

$\exists$ elliptic pt (compactness in $\mathbb{R}^3$) $\Rightarrow \exists$ hyp points + plane pts

2d) Let S_1 and S_2 be isometric surfaces in $\mathbf{R}^3$ where the second fundamental forms are same. Then there is a rigid motion mapping S_1 onto S_2.

TRUE: I and II same $\Rightarrow \exists$ rigid motion

Fund. Th. of Surfaces.

3) (20 pts) Construct a surface S which contains a point p such that the principal curvatures at p are 4 and -3. Compute the first and second fundamental form at p for S.

$z = ax^2 + by^2$ $\quad p:(0,0,0)$ $\quad \varphi: (x, y, ax^2+by^2)$

$\varphi_x = \langle 1, 0, 2ax \rangle$ $\quad E = 1+4a^2x^2$

$\varphi_y = \langle 0, 1, 2by \rangle$ $\quad F = 4abxy$ $\quad \Rightarrow I_{(0,0)} = x^2 + y^2$

$G = 1+4b^2y^2$ $\quad E=1 \; F=0 \; G=1$

$\varphi_{xx} = \langle 0,0,2a \rangle$ $\quad e = \varphi_{xx}\cdot N = 2a$

$\varphi_{yy} = \langle 0,0,2b \rangle$ $\Rightarrow$ $f = \varphi_{xy}\cdot N = 0$ $\quad \Rightarrow II_{(0,0)} = 2ax^2 + 2by^2$

$\varphi_{xy} = \langle 0,0,0 \rangle$ $\quad g = \varphi_{yy}\cdot N = 2b$

$N = \langle -2ax, -2by, 1 \rangle$

LA $\quad a = 2, \; b = -\frac{3}{2}$ $\quad \Rightarrow II_{(0,0)} = 4x^2 - 3y^2$

$\Rightarrow \max II\big|_{S^1} = 4$

$\min II\big|_{S^1} = -3$

$S: \left(z = 2x^2 - \frac{3}{2}y^2 \right)$ $\quad I_{(0,0)} = x^2 + y^2$

$p = (0,0,0)$ $\quad II_{(0,0)} = 4x^2 - 3y^2$

$K = \frac{eg - f^2}{EG - F^2}$

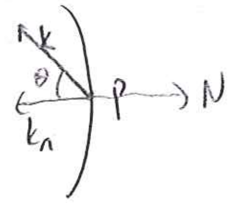
$= \frac{-4\cdot 3 \cdot 0}{1\cdot 1 - 0} = -12 = 4\cdot -3$ ✓

4) (20 pts) Let S be a round sphere with radius 2 in $\mathbb{R}^3$, and let $p \in S$. Show that there exists a curve $\alpha_k \subset S$ through p where its curvature at p is equal to k if and only if $k \geq \frac{1}{2}$.

$\Rightarrow$ $\alpha \subset S$ and S is sphere $\Rightarrow$ $k_n = \frac{1}{2}$ for any $\alpha \subset S$

$k = k_n^2 + k_g^2 \Rightarrow k \geq \frac{1}{2}$

$\Leftarrow$



$k_n = \frac{1}{2} = k\cos\theta \Rightarrow k = \frac{1}{2\cos\theta}$

let $\alpha_k = S \cap P_\theta$ where P_θ is the plane through p whose normal N_θ with $\angle(N, N_\theta) = \theta$.

$\Rightarrow$ $\forall k \geq \frac{1}{2}$ α_k for $\cos\theta = \frac{1}{2k} \leq 1$

$\alpha_k = S \cap P_\theta$ ✓

5a) (20 pts) Let α be regular curve in a surface S. Show that α is a line of curvature if and only if $N'(t) = \lambda(t)\alpha'(t)$ where $N(t) = N(\alpha(t))$ and $\lambda(t)$ is a differentiable function.

N normal to S. α line of curvature $\Rightarrow$

$\alpha'(t)$ is principal direction $\forall t$ $\Rightarrow$ eigenvector for dN

$\Rightarrow$ $N'(t) = dN(\alpha'(t)) = \lambda(t)\alpha'(t)$ $\square$

5b) Show that if $\alpha \subset S$ is both a line of curvature and a geodesic, then α is a plane curve.

let α is par. by arclgth.

α geodesic $\Rightarrow$ $k_g = 0$ & $k_n = k$ $\Rightarrow$ $\alpha'' \parallel N$

$\Rightarrow$ $\vec{N}_\alpha = \vec{N}_S$.

normal of curve

line of curvature $\Rightarrow$ $\vec{N}_S'(t) = \lambda(t)\alpha'(t)$. (by above)

$\Rightarrow$ $N_\alpha'(t) = \lambda(t)\underbrace{\alpha'(t)}_{T}$

but by Frenet frame $N' = kT + \tau B$

$\Rightarrow \tau = 0 \Rightarrow \alpha$ plane curve

6a) (20 pts) Show that the surfaces

$$\varphi(u, v) = (u\cos v, u \sin v, \log u) \qquad \psi(u, v) = (u \cos v, u \sin v, v)$$

have equal Gaussian curvature at the points $\varphi(u, v)$ and $\psi(u, v)$.

$K = \dfrac{eg - f^2}{EG - F^2}$

$\varphi_u = \langle \cos v, \sin v, \frac{1}{u} \rangle$ $\quad E = |\varphi_u|^2 = 1 + \frac{1}{u^2}$

$\varphi_v = \langle -u \sin v, u \cos v, 0 \rangle$ $\quad F = 0$

$G = u^2$

$\Rightarrow K = \dfrac{eg - f^2}{EG - F^2} = \dfrac{-\frac{1}{u} \cdot u - 0}{(1+\frac{1}{u^2}) u^2 - 0} = \dfrac{-1}{1+u^2}$

$\varphi_{uu} = \langle 0, 0, -\frac{1}{u^2} \rangle$ $\quad e = \varphi_{uu} \cdot N = \frac{-1}{u}$

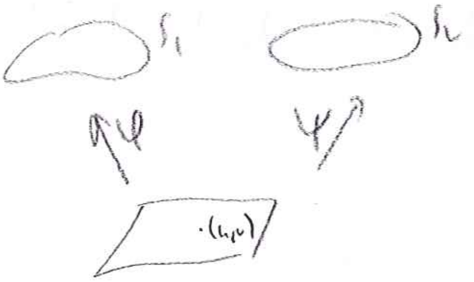
$\varphi_{uv} = \langle -\sin v, \cos v, 0 \rangle$ $\quad f = \varphi_{uv} \cdot N = 0$

$\varphi_{vv} = \langle -u \cos v, -u \sin v, 0 \rangle$ $\quad g = \varphi_{vv} \cdot N = u$

$N = \langle -\cos v, -\sin v, u \rangle$

Similarly compute $\psi(u,v)$ $\quad K = \dfrac{-1}{1+u^2}$

6b) Give a counterexample to the converse of the Gauss Theorem.
Hint: Show $\varphi \circ \psi^{-1}$ is not an isometry.



$E_\varphi = 1 + \frac{1}{u^2}$

$E_\psi = 1$

$\Rightarrow I_\varphi \neq I_\psi$

Gauss Theorem $\Rightarrow$ $\underset{\text{isometry}}{S_1 \simeq S_2} \Rightarrow K_{S_1} = K_{S_2}$

but converse not true

$K_{S_1} = K_{S_2} \not\Rightarrow S_1 \simeq S_2$