Chapter 3

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Chapter 3

3.1-1 Using Euler's identity, we have

$$\mathcal{F}\{g(t)\} = \int_{-\infty}^{\infty} g(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} g(t)\cos\omega t\,dt - j\int_{-\infty}^{\infty} g(t)\sin\omega t\,dt$$

If $g(t)$ is an even function of t, $g(t)\sin\omega t$ is an odd function of t, and the second integral vanishes. Moreover, $g(t)\cos\omega t$ is an even function of t, and the first integral is twice the integral over the interval 0 to ∞. Thus when $g(t)$ is even,

$$G(f) = 2\int_0^{\infty} g(t)\cos\omega t\,dt = 2\int_0^{\infty} g(t)\cos(2\pi ft)\,dt$$

Similar argument shows that when $g(t)$ is odd,

$$G(f) = -2j\int_0^{\infty} g(t)\sin 2\pi ft\,dt$$

If $g(t)$ is also real (in addition to being even), the integral from $G(f) = 2\int_0^{\infty} g(t)\cos\omega t\,dt$ real. Moreover,

$$G(-f) = 2\int_0^{\infty} g(t)\cos[2\pi(-f)t]\,dt == 2\int_0^{\infty} g(t)\cos(2\pi ft)\,dt = G(f)$$

Hence $G(f)$ is a real and even function of f. Similar arguments can be used to prove the rest of the properties.

3.1-2

(a)

$$\begin{aligned} G(f) &= \int_{-\infty}^{\infty} g(t)\,e^{-j2\pi ft}\,dt = \int_{-\infty}^{\infty} e^{-2|t-3|}e^{-j2\pi ft}\,dt \\ &= \int_{-\infty}^{3} e^{2(t-3)}e^{-j2\pi ft}\,dt + \int_{3}^{\infty} e^{-2(t-3)}e^{-j2\pi ft}\,dt \\ &= \frac{e^{-j6\pi f}}{2-j2\pi f} + \frac{e^{-j6\pi f}}{2+j2\pi f} = \frac{e^{-j6\pi f}}{1+\pi^2 f^2} \end{aligned}$$

(b)

$$g(t) = \int_{-\infty}^{\infty} G(f)\,e^{j2\pi ft}df = \int_{-\infty}^{\infty} \delta(4\pi f)\,e^{j2\pi ft}df - \int_{-\infty}^{\infty} \delta(f-2)\,e^{j2\pi ft}df$$

Let $\lambda = 4\pi f$ in the integral involving $\delta(4\pi f)$. Then

$$\int_{-\infty}^{\infty} \delta(4\pi f)\,e^{j2\pi ft}df = \int_{-\infty}^{\infty} \delta(\lambda)\,e^{j(\lambda/2)t}\frac{d\lambda}{4\pi} = \frac{e^0}{4\pi} = \frac{1}{4\pi}$$

We also have $\int_{-\infty}^{\infty} \delta(f-2)\,e^{j2\pi ft}df = e^{j4\pi t}$. Hence

$$g(t) = \frac{1}{4\pi} - e^{j4\pi t}$$

3.1-3

$$\mathcal{F}\{g^*(t)\} = \int_{-\infty}^{\infty} g^*(t)\,e^{-j2\pi ft}\,dt = \left(\int_{-\infty}^{\infty} g(t)\,e^{j2\pi ft}\,dt\right)^* = [G(-f)]^* \equiv G^*(-f)$$

3.1-4

(a)

$$G(f) = \int_0^1 4e^{-j2\pi ft}\,dt + \int_1^2 2e^{-j2\pi ft}\,dt = \frac{4 - 2e^{-j2\pi f} - 2e^{-j4\pi f}}{j2\pi f}$$

(b)

$$g(t) = \begin{cases} -t/T, & -\tau \le t \le 0 \\ t/T, & 0 \le t \le \tau \\ 0, & \text{else} \end{cases}$$

$$G(f) = \int_{-\tau}^{0} -\frac{t}{\tau}e^{-j2\pi ft}\,dt + \int_0^{\tau} \frac{t}{\tau}e^{-j2\pi ft}\,dt = \frac{2}{\tau(2\pi f)^2}[\cos 2\pi f\tau + 2\pi f\tau \sin 2\pi f\tau - 1]$$

This result could also be derived by observing that $g(t)$ is an even function. Therefore from the result in Problem 3.1-1,

$$G(f) = \frac{2}{\tau}\int_0^{\tau} t\cos 2\pi ft\,dt = \frac{2}{\tau(2\pi f)^2}[\cos 2\pi f\tau + 2\pi f\tau \sin 2\pi f\tau - 1]$$

3.1-5

(a) Because

$$G(f) = \begin{cases} 3f^2, & -B \le f \le B \\ 0, & \text{else} \end{cases}$$

is an even function of f,

$$\begin{aligned} g(t) &= \int_{-\infty}^{\infty} G(f)\,e^{j2\pi ft}\,df = 2\int_0^B 3f^2\cos 2\pi ft\,df \\ &= 6\left[\frac{2\pi ft\cos 2\pi ft + \left(2\pi^2 f^2 t^2 - 1\right)\sin 2\pi ft}{4\pi^3 t^3}\right]_{f=0}^{f=B} \\ &= 6\left[\frac{2\pi Bt\cos 2\pi Bt + \left(2\pi^2 B^2 t^2 - 1\right)\sin 2\pi Bt}{4\pi^3 t^3}\right] \end{aligned}$$

(b) Because

$$G(f) = \begin{cases} 1, & -2 \le f \le -1 \\ 2, & -1 \le f \le 1 \\ 1, & 1 \le f \le 2 \\ 0, & \text{else} \end{cases}$$

is an even function of f,

$$\begin{aligned} g(t) &= \int_{-\infty}^{\infty} G(f)\,e^{j2\pi ft}\,df = 2\int_0^1 2\cos 2\pi ft\,df + 2\int_1^2 \cos 2\pi ft\,df \\ &= \left.\frac{4\sin 2\pi ft}{2\pi t}\right|_{f=0}^{f=1} + \left.\frac{2\sin 2\pi ft}{2\pi t}\right|_{f=1}^{f=2} = \frac{\sin 2\pi t + \sin 4\pi t}{\pi t} \end{aligned}$$

3.2-1

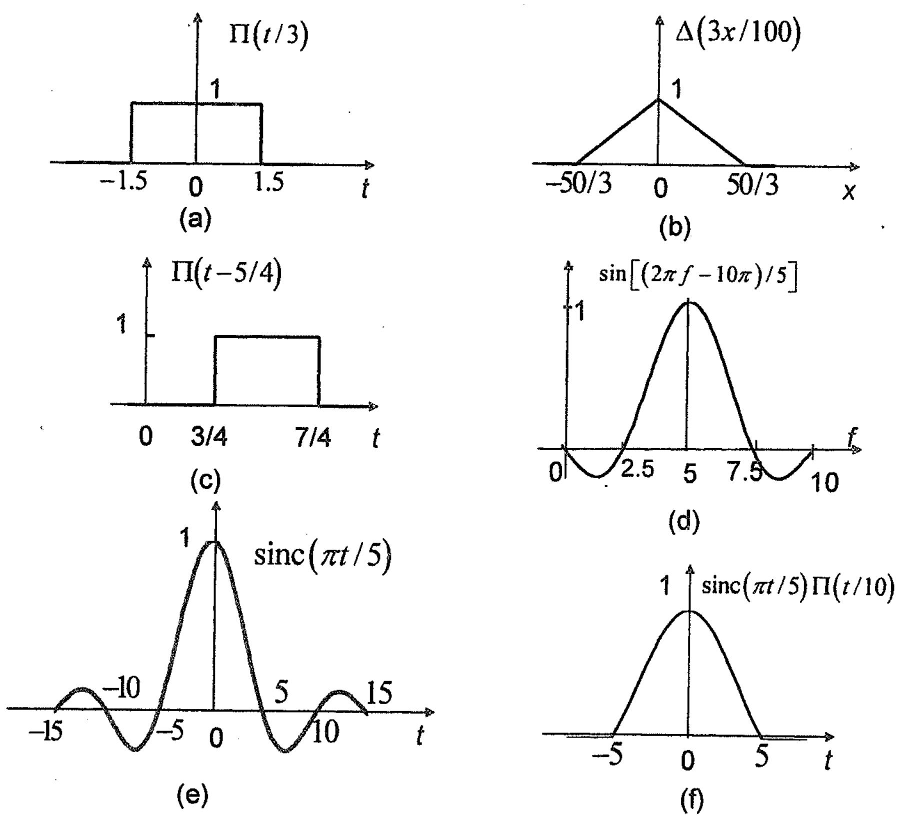

Fig. S3.2-1

(b) From Table 3.1, $2\cos 2\pi Tt \Longleftrightarrow \delta(f+T)+\delta(f-T)$. By duality, $\delta(t+T)+\delta(t-T) \Longleftrightarrow 2\cos(-2\pi fT) = 2\cos(2\pi fT)$.

(c) From Table 3.1, $2j\sin 2\pi Tt \Longleftrightarrow \delta(f-T)-\delta(f+T)$. By duality, $\delta(t-T)-\delta(t+T) \Longleftrightarrow 2j\sin(-2\pi fT) = -2j\sin(2\pi fT)$. Multiplying both sides by -1, we obtain the desired result.

3.3-2

(b) $g_1(t) = g(-t)$. Then $G_1(f) = \frac{1}{(2\pi f)^2}\left(e^{-j2\pi f} + j2\pi f e^{-j2\pi f} - 1\right)$

(c) $g_2(t) = g(t-1) + g_1(t-1)$. Then

$$\begin{aligned} G_2(f) &= \frac{e^{-j2\pi f}}{(2\pi f)^2}\left(e^{j2\pi f} - j2\pi f e^{j2\pi f} - 1\right) + \frac{e^{-j2\pi f}}{(2\pi f)^2}\left(e^{-j2\pi f} + j2\pi f e^{-j2\pi f} - 1\right) \\ &= \frac{1}{(2\pi f)^2}\left(1 - j2\pi f - 2e^{-j2\pi f} + e^{-j4\pi f} + j2\pi f e^{-j4\pi f}\right) \end{aligned}$$

(d) $g_3(t) = g(t-1) + g_1(t+1)$. Then

$$\begin{aligned} G_3(f) &= \frac{e^{-j2\pi f}}{(2\pi f)^2}\left(e^{j2\pi f} - j2\pi f e^{j2\pi f} - 1\right) + \frac{e^{j2\pi f}}{(2\pi f)^2}\left(e^{-j2\pi f} + j2\pi f e^{-j2\pi f} - 1\right) \\ &= \frac{1}{(2\pi f)^2}\left(2 - e^{-j2\pi f} - e^{j2\pi f}\right) = \frac{1}{2(\pi f)^2}\left(1 - 1\cos 2\pi f\right) \end{aligned}$$

(e) $g_4(t) = g\left(t - \frac{1}{2}\right) + g_1\left(t + \frac{1}{2}\right)$. Then

$$\begin{aligned} G_4(f) &= \frac{e^{-j\pi f}}{(2\pi f)^2}\left(e^{j2\pi f} - j2\pi f e^{j2\pi f} - 1\right) + \frac{e^{j\pi f}}{(2\pi f)^2}\left(e^{-j2\pi f} + j2\pi f e^{-j2\pi f} - 1\right) \\ &= \frac{1}{(2\pi f)^2}\left(-j2\pi f e^{j\pi f} + j2\pi f e^{-j\pi f}\right) = \frac{-j}{2\pi f}\left(e^{j\pi f} - e^{-j\pi f}\right) = \frac{1}{\pi f}\sin(\pi f) \end{aligned}$$

(f) $g_5(t) = 1.5g\left(\frac{1}{2}(t-2)\right)$. Then

$$G_5(f) = \frac{3e^{-j4\pi f}}{(4\pi f)^2}\left(e^{j4\pi f} - j4\pi f e^{j4\pi f} - 1\right) = \frac{3}{(4\pi f)^2}\left(1 - j4\pi f - e^{-j4\pi f}\right)$$

3.3-3

(a) $g(t) = \Pi\left(\frac{t + \frac{T}{2}}{T}\right) - \Pi\left(\frac{t - \frac{T}{2}}{T}\right)$. From pair 17 of Table 3.1 and the time-shifting property, we find

$$\Pi\left(\frac{t \pm \frac{T}{2}}{T}\right) \Longleftrightarrow T\,\text{sinc}(\pi f T)\, e^{\pm j\pi f T}$$

Then

$$G(f) = T\,\text{sinc}(\pi f T)\left[e^{j\pi f T} - e^{-j\pi f T}\right] = \frac{j2}{\pi f}\sin^2(\pi f T)$$

(b) $g(t) = \sin t\, u(t) + \sin(t-\pi)u(t-\pi)$. From pair 14 of Table 3.1 and the time-shifting property, we find

$$\sin t\, u(t) \Longleftrightarrow \frac{\pi}{4j}[\delta(f - 1/2\pi) - \delta(f + 1/2\pi)] + \frac{1}{1 - (2\pi f)^2}$$

and

$$\sin(t-\pi)u(t-\pi) \Longleftrightarrow \left\{\frac{\pi}{4j}[\delta(f - 1/2\pi) - \delta(f + 1/2\pi)] + \frac{1}{1 - (2\pi f)^2}\right\} e^{-j2\pi^2 f}$$

Then

$$G(f) = \left\{\frac{\pi}{4j}[\delta(f - 1/2\pi) - \delta(f + 1/2\pi)] + \frac{1}{1 - (2\pi f)^2}\right\}(1 + e^{-j2\pi^2 f})$$

Recall that $g(x)\delta(x - x_0) = g(x_0)\delta(x - x_0)$. Therefore $\delta(f \pm 1/2\pi)(1 + e^{-j2\pi^2 f}) = 0$, and

$$G(f) = \frac{1}{1 - (2\pi f)^2}(1 + e^{-j2\pi^2 f})$$

(c) $g(t) = \cos t\, u(t) + \sin\left(t - \frac{\pi}{2}\right) u\left(t - \frac{\pi}{2}\right)$. From pairs 13 and 14 of Table 3.1 and the time-shifting property, we find

$$\cos t\, u(t) \Longleftrightarrow \frac{\pi}{4}[\delta(f - 1/2\pi) - \delta(f + 1/2\pi)] + \frac{j2\pi f}{1 - (2\pi f)^2}$$

and

$$\sin(t - \frac{\pi}{2})u(t - \frac{\pi}{2}) \Longleftrightarrow \left\{\frac{\pi}{4j}[\delta(f - 1/2\pi) - \delta(f + 1/2\pi)] + \frac{1}{1 - (2\pi f)^2}\right\} e^{-j\pi^2 f}$$

Following Problem 3.3-3b and observing $\delta(f \pm 1/2\pi)e^{-j\pi^2 f} = \delta(f \pm 1/2\pi)e^{\pm j\pi/2} = \pm j\delta(f \pm 1/2\pi)$, we find

$$G(f) = \frac{j2\pi f}{1 - (2\pi f)^2} + \frac{e^{-j\pi^2 f}}{1 - (2\pi f)^2} = \frac{1}{1 - (2\pi f)^2}[j2\pi f + e^{-j\pi^2 f}]$$

(d) $g(t) = e^{-at}u(t) - e^{-aT}e^{-a(t-T)}u(t-T)$. From pair 1 of Table 3.1 and the time-shifting property, we find

$$G(f) = \frac{1}{a+j2\pi f} - \frac{e^{-aT}}{a+j2\pi f}e^{-j2\pi fT} = \frac{1}{a+j2\pi f}[1 - e^{-(a+j2\pi f)T}]$$

3.3-4 From the time-shifting property, $g(t \pm T) \Longleftrightarrow G(f)e^{\pm j2\pi fT}$. Therefore,

$$g(t+T) + g(t-T) \Longleftrightarrow G(f)e^{j2\pi fT} + G(f)e^{-j2\pi fT} = 2G(f)\cos 2\pi fT$$

(a) Let $T = 3$. Then $g(t) = \Pi\left(\frac{t}{2}\right)$ and from pair 17 of Table 3.1, $G(f) = 2\,\text{sinc}\,(2\pi f)$. Then

$$g(t+3) + g(t-3) \Longleftrightarrow 4\,\text{sinc}\,(2\pi f)\cos(6\pi f)$$

(b) Again, let $T = 3$ and then take $g(t) = \Delta\left(\frac{t}{2}\right)$. From pair 19 of Table 3.1, $G(f) = \text{sinc}^2(\pi f)$. Then

$$g(t+3) + g(t-3) \Longleftrightarrow 2\,\text{sinc}^2(\pi f)\cos(6\pi f)$$

3.3-5 To prove $g(t)\sin 2\pi f_0 t \Longleftrightarrow \frac{1}{2j}[G(f-f_0) - G(f+f_0)]$, we use Euler's identity to rewrite the sine function in terms of complex exponentials. Then we use the frequency-shifting property $g(t)e^{\pm j2\pi f_0 t} \Longleftrightarrow G(f \mp f_0)$ to obtain

$$g(t)\sin 2\pi f_0 t = \frac{1}{2j}[g(t)e^{j2\pi f_0 t} + g(t)e^{-j2\pi f_0 t}] \Longleftrightarrow \frac{1}{2j}[G(f-f_0) - G(f+f_0)]$$

To prove $\frac{1}{2j}[g(t+T) - g(t-T)] \Longleftrightarrow G(f)\sin 2\pi fT$, we employ the time-shifting property $g(t \pm T) \Longleftrightarrow G(f)e^{\pm j2\pi fT}$ to obtain

$$\frac{1}{2j}[g(t+T) - g(t-T)] \Longleftrightarrow \frac{1}{2j}\left[G(f)e^{j2\pi fT} - G(f)e^{-j2\pi fT}\right]$$

Using Euler's identity to recognize $\sin 2\pi fT$, we obtain the desired result.

To find the desired Fourier transform, we set $g(t) = \Pi\left(\frac{t}{2}\right)$ and $T = 3$. Then using pair 17 from Table 3.1 we obtain

$$g(t+3) - g(t-3) \Longleftrightarrow 4j\ \text{sinc}\,(2\pi f)\sin(6\pi f)$$

3.3-6 Note $\cos 2\pi f_0 t = \cos 10t \Rightarrow f_0 = 10/2\pi = 5/\pi$.

(a) We take $g(t) = \Delta(\frac{t}{2\pi})$. From pair 19 of Table 3.1 and the modulation property, we obtain

$$\Delta\left(\frac{t}{2\pi}\right)\cos 10t \Longleftrightarrow \frac{\pi}{2}\left[\text{sinc}^2(\pi^2 f - 5\pi) + \text{sinc}^2(\pi^2 f + 5\pi)\right]$$

(b) The signal $g(t)$ here is the same as the signal in Fig. S3.3-6 delayed by 2π. From the time-shifting property, its Fourier transform is the same as in part **(a)** multiplied by $e^{-j4\pi^2 f}$. Therefore,

$$G(f) = \frac{\pi}{2}\left[\text{sinc}^2(\pi^2 f - 5\pi) + \text{sinc}^2(\pi^2 f + 5\pi)\right]e^{-j4\pi^2 f}$$

The Fourier transform in this case is the same as that in part **(a)** multiplied by $e^{-j4\pi^2 f}$. This multiplying factor represents a linear phase spectrum $-4\pi^2 f$. Thus we have an amplitude spectrum [same as in part **(a)**] as well as a linear phase spectrum $\angle G(\omega) = -4\pi^2 f$ as shown in Fig. S3.3-6b.

Note: In the foregoing solution, we first multiplied the triangle pulse $\Delta(\frac{t}{2\pi})$ by $\cos 10t$ and then delayed the result by 2π. This means the signal in Fig. S3.3-6b is expressed as $\Delta(\frac{t-2\pi}{2\pi})\cos 10(t-2\pi)$. We could have interchanged the operation in this particular case; that is, we could have first delayed the triangle pulse $\Delta(\frac{t}{2\pi})$ by 2π and then multiplied the result by $\cos 10t$. In this alternate procedure, the signal in Fig. S3.3-6b would be expressed as

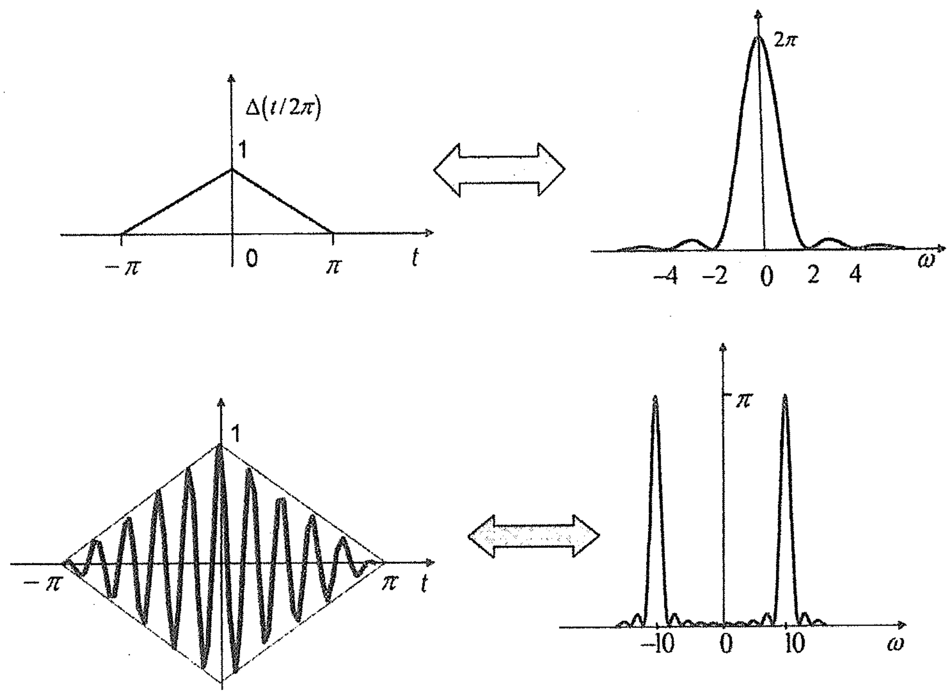

Fig. S3.3-6a

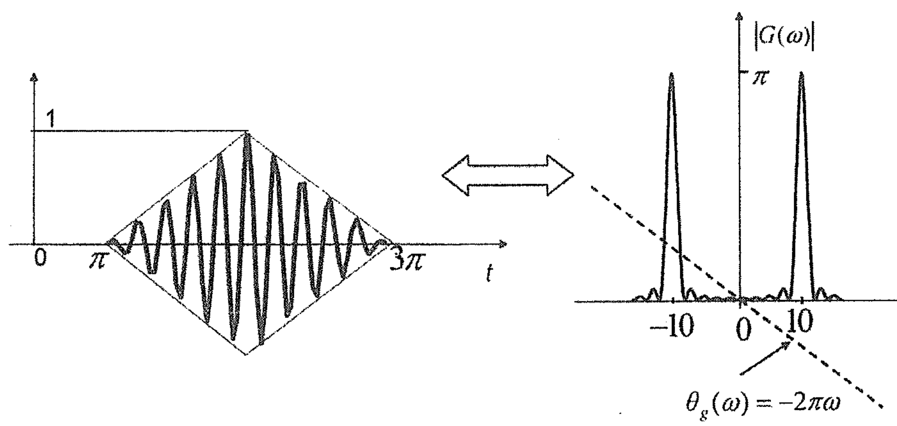

Fig. S3.3-6b

$\Delta(\frac{t-2\pi}{2\pi})\cos 10t$. This interchange of operation is permissible here only because the sinusoid $\cos 10t$ executes an integral number of cycles in the interval 2π. Thus both expressions are equivalent, since $\cos 10(t-2\pi) = \cos 10t$.

(c) In this case the signal is identical to that in Fig. S3.3-6b, except that the basic pulse is the gate pulse $\Pi\left(\frac{t}{2\pi}\right)$ instead of a triangle pulse $\Delta(\frac{t}{2\pi})$. From pair 19 of Table 3.1 and the modulation property, we obtain

$$G(f) = \frac{\pi}{2}\left[\text{sinc}\left(2\pi^2 f - 10\pi\right) + \text{sinc}\left(2\pi^2 f + 10\pi\right)\right] e^{-j4\pi^2 f}$$

3.3-7 $g(t)$ is band-limited to B Hz. Recall that the width property of convolution states that the bandwidth of two convolved signals is the sum of the bandwidths of the respective signals. Also recall the convolution property of the Fourier transform, that is,

$$g(t)\,g(t) \Longleftrightarrow G(f) * G(f)$$

Let $h(t) = g(t)\,g(t)$. The bandwidth of $h(t)$ is $2B$ Hz by the width property of convolution. We can apply the convolution property again and see

$$h(t)\,g(t) = g^3(t) \Longleftrightarrow G(f) * G(f) * G(f)$$

The resulting signal has bandwidth $2B + B = 3B$. By induction we find the bandwidth of $g^n(t)$ is nB Hz.

3.3-8

(a)

$$G(\omega) = \int_{-T}^{0} e^{-j\omega t}\,dt - \int_{0}^{T} e^{-j\omega t}\,dt = -\frac{1}{j\pi f}[1 - \cos 2\pi fT] = \frac{j2}{\pi f}\sin^2(\pi fT)$$

(b) $g(t) = \Pi\left(\frac{t+\frac{T}{2}}{T}\right) - \Pi\left(\frac{t-\frac{T}{2}}{T}\right)$.

From pair 17 of Table 3.1 and the time-shifting property, we find

$$\Pi\left(\frac{t \pm \frac{T}{2}}{T}\right) \Longleftrightarrow T\,\text{sinc}(\pi fT)\,e^{\pm j\pi fT}$$

Then

$$G(f) = T\,\text{sinc}(\pi fT)\left[e^{j\pi fT} - e^{-j\pi fT}\right] = \frac{j2}{\pi f}\sin^2(\pi fT)$$

(c) Recall $\frac{du(t+T)}{dt} = \delta(t+T)$. Then

$$\frac{dg(t)}{dt} = \delta(t+T) - 2\delta(t) + \delta(t-T)$$

By applying the time-differentiation and time-shifting properties, we see that the Fourier transform of the preceding equation yields

$$j2\pi f\,G(f) = e^{j2\pi fT} - 2 + e^{-j2\pi fT} = -2[1 - \cos 2\pi fT] = -4\sin^2(\pi fT)$$

and therefore

$$G(f) = \frac{j2}{\pi f}\sin^2(\pi fT)$$

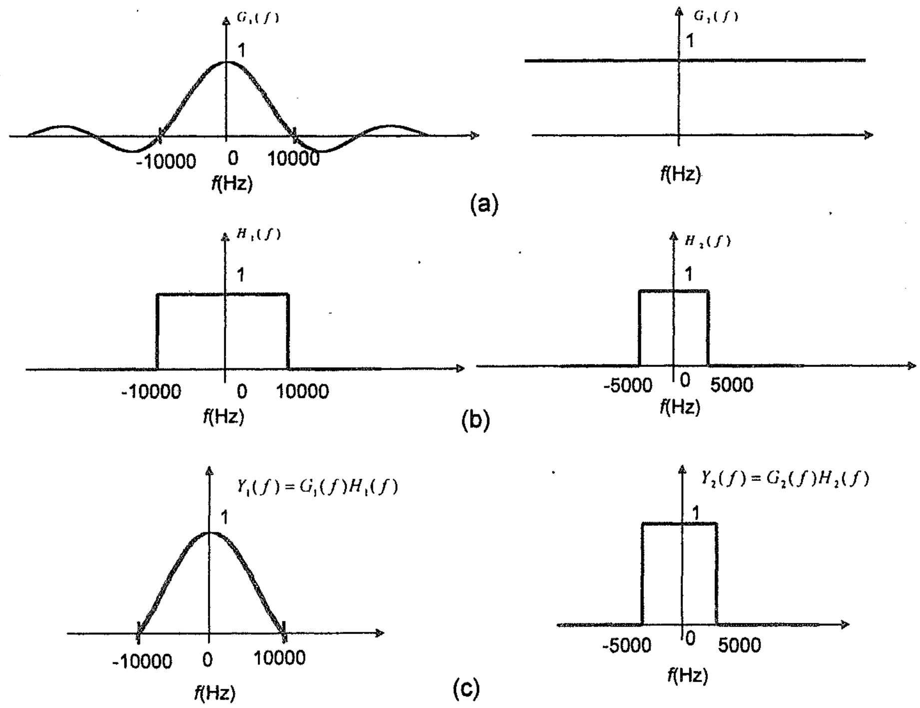

Fig. S3.4-2

(h) Let $x(t) = \alpha x_1(t) + \beta x_2(t)$ and $y_i(t) = \int_{-\infty}^{+\infty} Ap(t-\tau)\, x_i(\tau)\, d\tau$ for $i = 1, 2$. Then the system response to $x(t)$ is

$$\begin{aligned} y(t) &= \int_{-\infty}^{+\infty} Ap(t-\tau)\left[\alpha x_1(t) + \beta x_2(t)\right] d\tau \\ &= \alpha \int_{-\infty}^{+\infty} Ap(t-\tau)\, x_1(\tau)\, d\tau + \beta \int_{-\infty}^{+\infty} Ap(t-\tau)\, x_2(\tau)\, d\tau \\ &= \alpha y_1(t) + \beta y_2(t) \end{aligned}$$

and we have verified that the system is linear.

(i) Let $x(t) = \alpha x_1(t) + \beta x_2(t)$ (where α, β, $x_1(t)$, and $x_2(t)$ are complex), and $y_i(t) = Re\{x_i(t)\}$ for $i = 1, 2$. Then the system response to $x(t)$ is

$$\begin{aligned} y(t) &= Re\{\alpha x_1(t) + \beta x_2(t)\} \\ &= Re\{\alpha\}\, Re\{x_1(t)\} + Re\{\beta\}\, Re\{x_2(t)\} \\ &\neq \alpha y_1(t) + \beta y_2(t) = \alpha Re\{x_1(t)\} + \beta Re\{x_2(t)\} \end{aligned}$$

for complex α and β, so this system is not linear.

(j) By replacing Re with Im in part **(i)** we can easily verify that this system is not linear.

3.4-2

(a) See Fig. S3.4-2(a).

(b) See Fig. S3.4-2(b).

(c) See Fig. S3.4-2(c).

(d) We can read from the plots in Fig. S3.4-2 that the bandwidth of $y_1(t) = 10,000$ Hz and the bandwidth of $y_2(t) = 5000$ Hz. To find the bandwidth of their product $y(t)$, recall that multiplication in the time domain is equivalent to convolution in the frequency domain. Also recall the width property of convolution which states that the width of the convolution of two signals is equal to the sum of their respective widths. Then the bandwidth of $y(t) = 10,000 + 5000 = 15,000$ Hz.

3.4-3 To find the output $y(t)$ of this system, we will first find the Fourier transform $X(f)$ of the input. Since the Fourier transform is a linear operation, we can take the transform term by term. Using Table 3.1, we write

$$\begin{aligned} X(f) &= \delta(f) + 2e^{-j2\pi f t_0} - \frac{1}{2}\left[\delta\left(f + \frac{\omega_0}{2\pi}\right) + \delta\left(f - \frac{\omega_0}{2\pi}\right)\right] \\ &\quad + \sum_{i=1}^{n} A_i \frac{1}{a_i + j2\pi f} e^{-j2\pi f t_i} \end{aligned}$$

Then

$$\begin{aligned} Y(f) &= H(f)X(f) \\ &= \frac{1}{1+j2\pi f}\delta(f) + \frac{2}{1+j2\pi f}e^{-j2\pi f t_0} - \frac{1}{1+j2\pi f}\frac{1}{2}\left[\delta\left(f + \frac{\omega_0}{2\pi}\right) + \delta\left(f - \frac{\omega_0}{2\pi}\right)\right] \\ &\quad + \sum_{i=1}^{n} A_i \frac{1}{1+j2\pi f}\frac{1}{a_i + j2\pi f} e^{-j2\pi f t_i} \\ &= \delta(f) + \frac{2}{1+j2\pi f}e^{-j2\pi f t_0} - \frac{1}{2(1-j\omega_0)}\delta\left(f + \frac{\omega_0}{2\pi}\right) - \frac{1}{2(1+j\omega_0)}\delta\left(f - \frac{\omega_0}{2\pi}\right) \\ &\quad + \sum_{i=1}^{n} A_i \frac{1}{1+j2\pi f}\frac{1}{a_i + j2\pi f} e^{-j2\pi f t_i} \end{aligned}$$

From partial fraction expansion (assuming $a_i \neq 1$),

$$\frac{1}{1+j2\pi f}\frac{1}{a_i + j2\pi f} = \left(\frac{1}{a_i - 1}\right)\left(\frac{1}{1+j2\pi f} + \frac{1}{a_i + j2\pi f}\right)$$

If $a_i = 1$, then we have the term $\left(\frac{1}{1+j2\pi f}\right)^2$.

Taking the inverse Fourier transform term by term, we find

$$\begin{aligned} y(t) &= 1 + 2e^{-(t-t_0)}u(t-t_0) - \frac{1}{2(1-j\omega_0)}e^{-j\omega_0 t} - \frac{1}{2(1+j\omega_0)}e^{j\omega_0 t} \\ &\quad + \sum_{i=1,\, a_i \neq 1}^{n} \frac{A_i}{a_i - 1}\left(e^{-(t-t_0)} + e^{-a_i(t-t_0)}\right)u(t-t_0) \\ &\quad + \sum_{i=1,\, a_i = 1}^{n} A_i t e^{-(t-t_0)}u(t-t_0) \end{aligned}$$

3.5-1

(a) The step function $u(t)$ is 0 for $t < 0$, and so any function multiplied by it also has this property. Thus the system with this impulse response is causal.

(b) The indicated impulse response is symmetric and nonzero around $t = 0$, and therefore the corresponding system is not causal.

for t_0 (where, as in Problem 3.5-2, the impulse response is written explicitly as a function of t_0 in the numerator). We then take $h(t;t_0)\,u(t)$ as our approximate, realizable impulse response. In this case, we find

$$\int_{-\infty}^{\infty} |h(t)|^2\, dt = \int_{-\infty}^{\infty} e^{-2\beta|t-t_0|}\, dt = 2\int_0^{\infty} e^{-2\beta t}\, dt = \frac{1}{\beta}$$

and

$$\begin{aligned}\int_0^{\infty} |h(t;t_0)|^2\, dt &= \int_0^{\infty} e^{-2\beta|t-t_0|}\, dt = \int_0^{t_0} e^{2\beta(t-t_0)}\, dt + \int_{t_0}^{\infty} e^{-2\beta(t-t_0)}\, dt \\ &= \frac{1-e^{-2\beta t_0}}{2\beta} + \frac{1}{2\beta} = \frac{2-e^{-2\beta t_0}}{2\beta}\end{aligned}$$

Then we solve

$$\begin{aligned}\int_0^{\infty} |h(t;t_0)|^2\, dt &= 0.99\int_{-\infty}^{\infty} |h(t)|^2\, dt \\ \frac{2-e^{-2\beta t_0}}{2\beta} &= 0.99\frac{1}{\beta} \\ 2-e^{-2\beta t_0} &= 0.99\times 2 \\ \Rightarrow t_0 &= \frac{-\ln(2-0.99\times 2)}{2\beta}\end{aligned}$$

For example, for $\beta = \frac{1}{2}$, $t_0 = 3.912$ captures 99% of the signal energy. The resulting impulse response would be $h(t) = e^{-|t-3.912|/2}$.

3.5-4 For $RC = 10^{-3}$, the system's frequency response is

$$H(f) = \frac{1000}{1000 + j2\pi f}$$

Hence

$$|H(f)| = \frac{1000}{\sqrt{1000^2 + (2\pi f)^2}}$$

and

$$t_d(f) = \frac{1000}{(2\pi f)^2 + 1000^2}$$

Observe from Figure 3.28 that both the amplitude response and the time delay of the circuit are monotonically decreasing functions of frequency. This monotonicity allows us to check the endpoints of the bands over which the signal $g(t)$ is nonzero to determine whether transmission is distortionless. These frequencies are $f_l = 99{,}000$ Hz and $f_h = 101{,}000$ Hz, and

$$\begin{aligned}|H(f_l)| &= 1.607\times 10^{-3} \qquad t_d(f_l) = 2.58\times 10^{-9} \\ |H(f_h)| &= 1.576\times 10^{-3} \qquad t_d(f_h) = 2.48\times 10^{-9}\end{aligned}$$

We compute the variations as

$$\begin{aligned}\Delta|H(f)| &= \frac{||H(f_h)| - |H(f_l)||}{\frac{1}{2}(|H(f_h)| + |H(f_l)|)}\times 100\% = 1.95\% < 2\% \\ \Delta t_d(f) &= \frac{|t_d(f_h) - t_d(f_l)|}{\frac{1}{2}(t_d(f_h) + t_d(f_l))}\times 100\% = 3.9\% > 1\%\end{aligned}$$

The variations are NOT within tolerances. Thus, we cannot consider the transmission to be distortionless. Still, if we wish to make an approximate, the average amplitude response is approximately $\frac{1}{2}(|H(f_h)| + |H(f_l)|) = 1.59\times 10^{-3}$, and the average delay is $\frac{1}{2}(t_d(f_h) + t_d(f_l)) = 6.14\times 10^{-4}$. The resulting approximate output signal is $y(t) \approx 1.59\times 10^{-3} g\left(t - 6.14\times 10^{-4}\right)$.

3.6-1

3.7-2 Recall that

$$g_2(t) = \int_{-\infty}^{\infty} G_2(f)e^{j2\pi ft}\,df \quad \text{and} \quad \int_{-\infty}^{\infty} g_1(t)e^{j2\pi ft}\,dt = G_1(-f)$$

Therefore

$$\begin{aligned}\int_{-\infty}^{\infty} g_1(t)g_2(t)\,dt &= \int_{-\infty}^{\infty} g_1(t)\left[\int_{-\infty}^{\infty} G_2(f)e^{j2\pi ft}\,df\right]dt \\ &= \int_{-\infty}^{\infty} G_2(f)\left[\int_{-\infty}^{\infty} g_1(t)e^{j2\pi ft}\,df\right]df \\ &= \int_{-\infty}^{\infty} G_1(-f)G_2(f)df\end{aligned}$$

Interchanging the roles of $g_1(t)$ and $g_2(t)$ in the preceding development, we can similarly show that

$$\int_{-\infty}^{\infty} g_1(t)g_2(t)\,dt = \int_{-\infty}^{\infty} G_1(f)G_2(-f)df$$

3.7-3 Take $g_1(t) = \text{sinc}(2\pi Bt - m\pi)$ and $g_2(t) = \text{sinc}(2\pi Bt - n\pi)$. Employing the hint, we find $G_1(f) = \frac{1}{2B}\Pi\left(\frac{f}{2B}\right)e^{-j\pi fm/B}$ and $G_2(f) = \frac{1}{2B}\Pi\left(\frac{f}{2B}\right)e^{-j\pi fn/B}$. Applying the result

$$\int_{-\infty}^{\infty} g_1(t)g_2(t)\,dt = \int_{-\infty}^{\infty} G_1(f)G_2(-f)df$$

from Problem 3.7-2, we find

$$\begin{aligned}\int_{-\infty}^{\infty} g_1(t)g_2(t)\,dt &= \int_{-\infty}^{\infty} \text{sinc}(2\pi Bt - m\pi)\,\text{sinc}(2\pi Bt - n\pi)\,dt \\ &= \int_{-\infty}^{\infty} G_1(f)G_2(-f)df \\ &= \int_{-\infty}^{\infty}\left[\frac{1}{2B}\Pi\left(\frac{f}{2B}\right)e^{-j\pi fm/B}\right]\left[\frac{1}{2B}\Pi\left(\frac{f}{2B}\right)e^{j\pi fn/B}\right]df \\ &= \frac{1}{4B^2}\int_{-B}^{B} e^{j[(n-m)/2B]2\pi f}\,df\end{aligned}$$

where the last line was obtained by observing that $\Pi\left(\frac{f}{2B}\right)$ effectively limits the integration interval to $\pm B$ and that we can multiply and divide the exponent of e by a factor of 2 without affecting the integral. To show the desired result, let $l = n - m$. Then

$$\begin{aligned}\frac{1}{4B^2}\int_{-B}^{B} e^{j[(n-m)/2B]2\pi f}\,df &= \frac{1}{4B^2}\left[\frac{e^{j(l/2B)2\pi f}}{j\,(l/2B)\,2\pi}\right]_{-B}^{B} \\ &= \frac{1}{4B^2}\left[\frac{e^{jl\pi}}{jl\pi/B} - \frac{e^{-jl\pi}}{jl\pi/B}\right] \\ &= \frac{1}{4B^2}\frac{2\sin\pi l}{l\pi/B} = \frac{\sin\pi l}{2Bl} \\ &= \begin{cases} 0, & l \in \{\pm1, \pm2, \pm3, \cdots\} \\ \frac{1}{2B}, & l = 0 \end{cases}\end{aligned}$$

To verify the foregoing, observe that for $l \neq 0$, $\sin\pi l = 0$ and the denominator is finite, so the entire term is zero. In the case $l = 0$,we have $\sin 0/0 = 1$ from L'Hôpital's rule, and the term becomes $\frac{1}{2B}$. Note that $l = 0$ implies $m = n$ and $l \neq 0$ implies $m \neq n$.