Problem 2.2.3 Solution

(a) We must choose c to make the PMF of V sum to one.

$$\sum_{v=1}^{4} P_V(v) = c(1^2 + 2^2 + 3^2 + 4^2) = 30c = 1 \tag{1}$$

Hence $c = 1/30$.

(b) Let $U = \{u^2 | u = 1, 2, \ldots\}$ so that

$$P[V \in U] = P_V(1) + P_V(4) = \frac{1}{30} + \frac{4^2}{30} = \frac{17}{30} \tag{2}$$

(c) The probability that V is even is

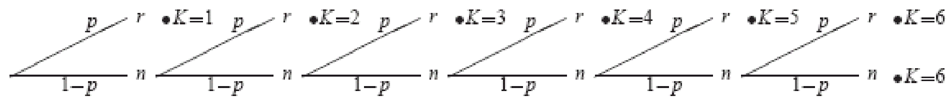
$$P[V \text{ is even}] = P_V(2) + P_V(4) = \frac{2^2}{30} + \frac{4^2}{30} = \frac{2}{3} \tag{3}$$

(d) The probability that $V > 2$ is

$$P[V > 2] = P_V(3) + P_V(4) = \frac{3^2}{30} + \frac{4^2}{30} = \frac{5}{6} \tag{4}$$

Problem 2.2.9 Solution

(a) In the setup of a mobile call, the phone will send the "SETUP" message up to six times. Each time the setup message is sent, we have a Bernoulli trial with success probability p. Of course, the phone stops trying as soon as there is a success. Using r to denote a successful response, and n a non-response, the sample tree is



(b) We can write the PMF of K, the number of "SETUP" messages sent as

$$P_K(k) = \begin{cases} (1-p)^{k-1}p & k = 1, 2, \ldots, 5 \\ (1-p)^5 p + (1-p)^6 = (1-p)^5 & k = 6 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

Note that the expression for $P_K(6)$ is different because $K = 6$ if either there was a success or a failure on the sixth attempt. In fact, $K = 6$ whenever there were failures on the first five attempts which is why $P_K(6)$ simplifies to $(1-p)^5$.

(c) Let B denote the event that a busy signal is given after six failed setup attempts. The probability of six consecutive failures is $P[B] = (1-p)^6$. To be sure that $P[B] \le 0.02$, we need $p \ge 1 - (0.02)^{1/6} = 0.479$.

Problem 2.3.7 Solution

Since an average of $T/5$ buses arrive in an interval of T minutes, buses arrive at the bus stop at a rate of $1/5$ buses per minute.

(a) From the definition of the Poisson PMF, the PMF of B, the number of buses in T minutes, is

$$P_B(b) = \begin{cases} (T/5)^b e^{-T/5}/b! & b = 0, 1, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(b) Choosing $T = 2$ minutes, the probability that three buses arrive in a two minute interval is

$$P_B(3) = (2/5)^3 e^{-2/5}/3! \approx 0.0072 \tag{2}$$

(c) By choosing $T = 10$ minutes, the probability of zero busees arriving in a ten minute interval is

$$P_B(0) = e^{-10/5}/0! = e^{-2} \approx 0.135 \tag{3}$$

(d) The probability that at least one bus arrives in T minutes is

$$P[B \geq 1] = 1 - P[B = 0] = 1 - e^{-T/5} \geq 0.99 \tag{4}$$

Rearranging yields $T \geq 5 \ln 100 \approx 23.0$ minutes.

Problem 2.3.9 Solution

The quirement that $\sum_{x=1}^{n} P_X(x) = 1$ implies

$$n = 1: \qquad c(1)\left[\frac{1}{1}\right] = 1 \qquad c(1) = 1 \tag{1}$$

$$n = 2: \qquad c(2)\left[\frac{1}{1} + \frac{1}{2}\right] = 1 \qquad c(2) = \frac{2}{3} \tag{2}$$

$$n = 3: \qquad c(3)\left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3}\right] = 1 \qquad c(3) = \frac{6}{11} \tag{3}$$

$$n = 4: \qquad c(4)\left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right] = 1 \qquad c(4) = \frac{12}{25} \tag{4}$$

$$n = 5: \qquad c(5)\left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}\right] = 1 \qquad c(5) = \frac{12}{25} \tag{5}$$

$$n = 6: \qquad c(6)\left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6}\right] = 1 \qquad c(6) = \frac{20}{49} \tag{6}$$

As an aside, find $c(n)$ for large values of n is easy using the recursion

$$\frac{1}{c(n+1)} = \frac{1}{c(n)} + \frac{1}{n+1}. \tag{7}$$

Problem 2.4.8 Solution

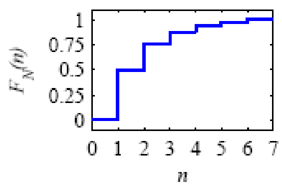
From Problem 2.2.9, the PMF of the number of call attempts is

$$P_N(n) = \begin{cases} (1-p)^{k-1}p & k = 1, 2, \ldots, 5 \\ (1-p)^5 p + (1-p)^6 = (1-p)^5 & k = 6 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

For $p = 1/2$, the PMF can be simplified to

$$P_N(n) = \begin{cases} (1/2)^n & n = 1, 2, \ldots, 5 \\ (1/2)^5 & n = 6 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

The corresponding CDF of N is



$$F_N(n) = \begin{cases} 0 & n < 1 \\ 1/2 & 1 \le n < 2 \\ 3/4 & 2 \le n < 3 \\ 7/8 & 3 \le n < 4 \\ 15/16 & 4 \le n < 5 \\ 31/32 & 5 \le n < 6 \\ 1 & n \ge 6 \end{cases} \quad (3)$$

Problem 2.5.2 Solution

Voice calls and data calls each cost 20 cents and 30 cents respectively. Furthermore the respective probabilities of each type of call are 0.6 and 0.4.

(a) Since each call is either a voice or data call, the cost of one call can only take the two values associated with the cost of each type of call. Therefore the PMF of X is

$$P_X(x) = \begin{cases} 0.6 & x = 20 \\ 0.4 & x = 30 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(b) The expected cost, $E[C]$, is simply the sum of the cost of each type of call multiplied by the probability of such a call occurring.

$$E[C] = 20(0.6) + 30(0.4) = 24 \text{ cents} \tag{2}$$

Problem 2.6.6 Solution

The cellular calling plan charges a flat rate of \$20 per month up to and including the 30th minute, and an additional 50 cents for each minute over 30 minutes. Knowing that the time you spend on the phone is a geometric random variable M with mean $1/p = 30$, the PMF of M is

$$P_M(m) = \begin{cases} (1-p)^{m-1}p & m = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The monthly cost, C obeys

$$P_C(20) = P[M \le 30] = \sum_{m=1}^{30} (1-p)^{m-1}p = 1 - (1-p)^{30} \tag{2}$$

When $M \ge 30$, $C = 20 + (M - 30)/2$ or $M = 2C - 10$. Thus,

$$P_C(c) = P_M(2c - 10) \qquad c = 20.5, 21, 21.5, \ldots \tag{3}$$

The complete PMF of C is

$$P_C(c) = \begin{cases} 1 - (1-p)^{30} & c = 20 \\ (1-p)^{2c-10-1}p & c = 20.5, 21, 21.5, \ldots \end{cases} \tag{4}$$

Problem 2.7.6 Solution

Since our phone use is a geometric random variable M with mean value $1/p$,

$$P_M(m) = \begin{cases} (1-p)^{m-1}p & m = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

For this cellular billing plan, we are given no free minutes, but are charged half the flat fee. That is, we are going to pay 15 dollars regardless and \$1 for each minute we use the phone. Hence $C = 15 + M$ and for $c \geq 16$, $P[C = c] = P[M = c - 15]$. Thus we can construct the PMF of the cost C

$$P_C(c) = \begin{cases} (1-p)^{c-16}p & c = 16, 17, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Since $C = 15 + M$, the expected cost per month of the plan is

$$E[C] = E[15 + M] = 15 + E[M] = 15 + 1/p \tag{3}$$

In Problem 2.7.5, we found that that the expected cost of the plan was

$$E[C] = 20 + \frac{(1-p)^{30}}{2p} \tag{4}$$

In comparing the expected costs of the two plans, we see that the new plan is better (i.e. cheaper) if

$$15 + 1/p \leq 20 + \frac{(1-p)^{30}}{2p} \tag{5}$$

A simple plot will show that the new plan is better if $p \leq p_0$ where $p_0 \approx 0.2$.

Problem 2.8.9 Solution

With our measure of jitter being σ_T, and the fact that $T = 2X - 1$, we can express the jitter as a function of q by realizing that

$$\text{Var}[T] = 4\,\text{Var}[X] = \frac{4q}{(1-q)^2} \tag{1}$$

Therefore, our maximum permitted jitter is

$$\sigma_T = \frac{2\sqrt{q}}{(1-q)} = 2 \text{ msec} \tag{2}$$

Solving for q yields $q^2 - 3q + 1 = 0$. By solving this quadratic equation, we obtain

$$q = \frac{3 \pm \sqrt{5}}{2} = 3/2 \pm \sqrt{5}/2 \tag{3}$$

Since q must be a value between 0 and 1, we know that a value of $q = 3/2 - \sqrt{5}/2 \approx 0.382$ will ensure a jitter of at most 2 milliseconds.

Problem 2.9.6 Solution

(a) Consider each circuit test as a binomial trial such that a failed circuit is called a success. The number of trias until the first success (i.e. a failed circuit) has the geometric PMF

$$P_N(n) = \begin{cases} (1-p)^{n-1}p & n = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(b) The probability there are at least 20 tests is

$$P[B] = P[N \geq 20] = \sum_{n=20}^{\infty} P_N(n) = (1-p)^{19} \tag{2}$$

Note that $(1-p)^{19}$ is just the probability that the first 19 circuits pass the test, which is what we would expect since there must be at least 20 tests if the first 19 circuits pass. The conditional PMF of N given B is

$$P_{N|B}(n) = \begin{cases} \frac{P_N(n)}{P[B]} & n \in B \\ 0 & \text{otherwise} \end{cases} = \begin{cases} (1-p)^{n-20}p & n = 20, 21, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

(c) Given the event B the conditional expectation of N is

$$E[N|B] = \sum_n n P_{N|B}(n) = \sum_{n=20}^{\infty} n(1-p)^{n-20}p \tag{4}$$

Making the substitution $j = n - 19$ yields

$$E[N|B] = \sum_{j=1}^{\infty} (j+19)(1-p)^{j-1}p = 1/p + 19 \tag{5}$$

We see that in the above sum, we effectively have the expected value of $J + 19$ where J is geometric random variable with parameter p. This is not surprising since the $N \geq 20$ iff we observed 19 successful tests. After 19 successful tests, the number of additional tests needed to find the first failure is still a geometric random variable with mean $1/p$.