Problem 3.1.1 Solution

The CDF of X is

$$F_X(x) = \begin{cases} 0 & x < -1 \\ (x+1)/2 & -1 \le x < 1 \\ 1 & x \ge 1 \end{cases} \tag{1}$$

Each question can be answered by expressing the requested probability in terms of $F_X(x)$.

(a)

$$P[X > 1/2] = 1 - P[X \le 1/2] = 1 - F_X(1/2) = 1 - 3/4 = 1/4 \tag{2}$$

(b) This is a little trickier than it should be. Being careful, we can write

$$P[-1/2 \le X < 3/4] = P[-1/2 < X \le 3/4] + P[X = -1/2] - P[X = 3/4] \tag{3}$$

Since the CDF of X is a continuous function, the probability that X takes on any specific value is zero. This implies $P[X = 3/4] = 0$ and $P[X = -1/2] = 0$. (If this is not clear at this point, it will become clear in Section 3.6.) Thus,

$$P[-1/2 \le X < 3/4] = P[-1/2 < X \le 3/4] = F_X(3/4) - F_X(-1/2) = 5/8 \tag{4}$$

(c)

$$P[|X| \le 1/2] = P[-1/2 \le X \le 1/2] = P[X \le 1/2] - P[X < -1/2] \tag{5}$$

Note that $P[X \le 1/2] = F_X(1/2) = 3/4$. Since the probability that $P[X = -1/2] = 0$, $P[X < -1/2] = P[X \le 1/2]$. Hence $P[X < -1/2] = F_X(-1/2) = 1/4$. This implies

$$P[|X| \le 1/2] = P[X \le 1/2] - P[X < -1/2] = 3/4 - 1/4 = 1/2 \tag{6}$$

(d) Since $F_X(1) = 1$, we must have $a \le 1$. For $a \le 1$, we need to satisfy

$$P[X \le a] = F_X(a) = \frac{a+1}{2} = 0.8 \tag{7}$$

Thus $a = 0.6$.

Problem 3.2.4 Solution

For $x < 0$, $F_X(x) = 0$. For $x \geq 0$,

$$F_X(x) = \int_0^x f_X(y)\,dy = \int_0^x a^2 y e^{-a^2y^2/2}\,dy = -e^{-a^2y^2/2}\Big|_0^x = 1 - e^{-a^2x^2/2} \quad (1)$$

A complete expression for the CDF of X is

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-a^2x^2/2} & x \geq 0 \end{cases} \quad (2)$$

Problem 3.3.1 Solution

$$f_X(x) = \begin{cases} 1/4 & -1 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

We recognize that X is a uniform random variable from [-1,3].

(a) $E[X] = 1$ and $\text{Var}[X] = \frac{(3+1)^2}{12} = 4/3$.

(b) The new random variable Y is defined as $Y = h(X) = X^2$. Therefore

$$h(E[X]) = h(1) = 1 \tag{2}$$

and

$$E[h(X)] = E\left[X^2\right] = \text{Var}[X] + E[X]^2 = 4/3 + 1 = 7/3 \tag{3}$$

Finally

$$E[Y] = E[h(X)] = E[X^2] = 7/3 \tag{4}$$

$$\text{Var}[Y] = E[X^4] - E[X^2]^2 = \int_{-1}^{3} \frac{x^4}{4}\,dx - \frac{49}{9} = \frac{61}{5} - \frac{49}{9} \tag{5}$$

Problem 3.3.7 Solution

To find the moments, we first find the PDF of U by taking the derivative of $F_U(u)$. The CDF and corresponding PDF are

$$F_U(u) = \begin{cases} 0 & u < -5 \\ (u+5)/8 & -5 \le u < -3 \\ 1/4 & -3 \le u < 3 \\ 1/4 + 3(u-3)/8 & 3 \le u < 5 \\ 1 & u \ge 5. \end{cases} \qquad f_U(u) = \begin{cases} 0 & u < -5 \\ 1/8 & -5 \le u < -3 \\ 0 & -3 \le u < 3 \\ 3/8 & 3 \le u < 5 \\ 0 & u \ge 5. \end{cases} \tag{1}$$

(a) The expected value of U is

$$E[U] \int_{-\infty}^{\infty} u f_U(u)\, du = \int_{-5}^{-3} \frac{u}{8}\, du + \int_{3}^{5} \frac{3u}{8}\, du \tag{2}$$

$$= \left.\frac{u^2}{16}\right|_{-5}^{-3} + \left.\frac{3u^2}{16}\right|_{3}^{5} \tag{3}$$

$$= -1 + 3 = 2 \tag{4}$$

(b) The second moment of U is

$$E[U^2] \int_{-\infty}^{\infty} u^2 f_U(u)\, du = \int_{-5}^{-3} \frac{u^2}{8}\, du + \int_{3}^{5} \frac{3u^2}{8}\, du \tag{5}$$

$$= \left.\frac{u^3}{24}\right|_{-5}^{-3} + \left.\frac{u^3}{8}\right|_{3}^{5} \tag{6}$$

$$= 49/3 \tag{7}$$

The variance of U is $\text{Var}[U] = E[U^2] - (E[U])^2 = 37/3$.

(c) Note that $2^U = e^{(\ln 2)U}$. This implies that

$$\int 2^u\, du = \int e^{(\ln 2)u}\, du = \frac{1}{\ln 2} e^{(\ln 2)u} = \frac{2^u}{\ln 2} \tag{8}$$

The expected value of 2^U is then

$$E[2^U] = \int_{-\infty}^{\infty} 2^u f_U(u)\, du = \int_{-5}^{-3} \frac{2^u}{8}\, du + \int_{3}^{5} \frac{3 \cdot 2^u}{8}\, du \tag{9}$$

$$= \left.\frac{2^u}{8 \ln 2}\right|_{-5}^{-3} + \left.\frac{3 \cdot 2^u}{8 \ln 2}\right|_{3}^{5} \tag{10}$$

$$= \frac{2307}{256 \ln 2} = 13.001 \tag{11}$$

Problem 3.4.1 Solution

The reflecte power Y has an exponential ($\lambda = 1/P_0$) PDF. From Theorem 3.8, $E[Y] = P_0$. The probability that an aircraft is correctly identified is

$$P[Y > P_0] = \int_{P_0}^{\infty} \frac{1}{P_0} e^{-y/P_0}\, dy = e^{-1}. \tag{1}$$

Fortunately, real radar systems offer better performance.

Problem 3.4.10 Solution

The integral I_1 is

$$I_1 = \int_0^\infty \lambda e^{-\lambda x}\, dx = -e^{-\lambda x}\Big|_0^\infty = 1 \tag{1}$$

For $n > 1$, we have

$$I_n = \int_0^\infty \underbrace{\frac{\lambda^{n-1} x^{n-1}}{(n-1)!}}_{u} \underbrace{\lambda e^{-\lambda x}\, dt}_{dv} \tag{2}$$

We define u and dv as shown above in order to use the integration by parts formula $\int u\, dv = uv - \int v\, du$. Since

$$du = \frac{\lambda^{n-1} x^{n-1}}{(n-2)!}\, dx \qquad v = -e^{-\lambda x} \tag{3}$$

we can write

$$\begin{aligned} I_n &= uv|_0^\infty - \int_0^\infty v\, du && (4)\\ &= -\frac{\lambda^{n-1} x^{n-1}}{(n-1)!} e^{-\lambda x}\bigg|_0^\infty + \int_0^\infty \frac{\lambda^{n-1} x^{n-1}}{(n-2)!} e^{-\lambda x}\, dx && (5)\\ &= 0 + I_{n-1} && (6) \end{aligned}$$

Hence, $I_n = 1$ for all $n \geq 1$.

Problem 3.4.5 Solution

(a) The PDF of a continuous uniform random variable distributed from $[-5, 5)$ is

$$f_X(x) = \begin{cases} 1/10 & -5 \leq x \leq 5 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(b) For $x < -5$, $F_X(x) = 0$. For $x \geq 5$, $F_X(x) = 1$. For $-5 \leq x \leq 5$, the CDF is

$$F_X(x) = \int_{-5}^{x} f_X(\tau)\, d\tau = \frac{x+5}{10} \tag{2}$$

The complete expression for the CDF of X is

$$F_X(x) = \begin{cases} 0 & x < -5 \\ (x+5)/10 & 5 \leq x \leq 5 \\ 1 & x > 5 \end{cases} \tag{3}$$

(c) the expected value of X is

$$\int_{-5}^{5} \frac{x}{10}\, dx = \left. \frac{x^2}{20} \right|_{-5}^{5} = 0 \tag{4}$$

Another way to obtain this answer is to use Theorem 3.6 which says the expected value of X is

$$E[X] = \frac{5 + -5}{2} = 0 \tag{5}$$

(d) The fifth moment of X is

$$\int_{-5}^{5} \frac{x^5}{10}\, dx = \left. \frac{x^6}{60} \right|_{-5}^{5} = 0 \tag{6}$$

(e) The expected value of e^X is

$$\int_{-5}^{5} \frac{e^x}{10}\, dx = \left. \frac{e^x}{10} \right|_{-5}^{5} = \frac{e^5 - e^{-5}}{10} = 14.84 \tag{7}$$

Problem 3.5.10 Solution

This problem is mostly calculus and only a little probability. From the problem statement, the SNR Y is an exponential $(1/\gamma)$ random variable with PDF

$$f_Y(y) = \begin{cases} (1/\gamma)e^{-y/\gamma} & y \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Thus, from the problem statement, the BER is

$$\overline{P}_e = E[P_e(Y)] = \int_{-\infty}^{\infty} Q(\sqrt{2y}) f_Y(y)\, dy = \int_0^{\infty} Q(\sqrt{2y}) \frac{y}{\gamma} e^{-y/\gamma}\, dy \tag{2}$$

Like most integrals with exponential factors, its a good idea to try integration by parts. Before doing so, we recall that if X is a Gaussian (0, 1) random variable with CDF $F_X(x)$, then

$$Q(x) = 1 - F_X(x). \tag{3}$$

It follows that $Q(x)$ has derivative

$$Q'(x) = \frac{dQ(x)}{dx} = -\frac{dF_X(x)}{dx} = -f_X(x) = -\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \tag{4}$$

To solve the integral (2), we use the integration by parts formula $\int_a^b u\, dv = uv|_a^b - \int_a^b v\, du$, where

$$u = Q(\sqrt{2y}) \qquad dv = \frac{1}{\gamma} e^{-y/\gamma}\, dy \tag{5}$$

$$du = Q'(\sqrt{2y}) \frac{1}{\sqrt{2y}} = -\frac{e^{-y}}{2\sqrt{\pi y}} \qquad v = -e^{-y/\gamma} \tag{6}$$

From integration by parts, it follows that

$$\overline{P}_e = uv|_0^{\infty} - \int_0^{\infty} v\, du \tag{7}$$

$$= -Q(\sqrt{2y}) e^{-y/\gamma} \Big|_0^{\infty} - \int_0^{\infty} \frac{1}{\sqrt{y}} e^{-y[1+(1/\gamma)]}\, dy \tag{8}$$

$$= 0 + Q(0)e^{-0} - \frac{1}{2\sqrt{\pi}} \int_0^{\infty} y^{-1/2} e^{-y/\bar{\gamma}}\, dy \tag{9}$$

where $\bar{\gamma} = \gamma/(1+\gamma)$. Next, recalling that $Q(0) = 1/2$ and making the substitution $t = y/\bar{\gamma}$, we obtain

$$\overline{P}_e = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\bar{\gamma}}{\pi}} \int_0^{\infty} t^{-1/2} e^{-t}\, dt \tag{10}$$

From Math Fact B.11, we see that the remaining integral is the $\Gamma(z)$ function evaluated $z = 1/2$. Since $\Gamma(1/2) = \sqrt{\pi}$,

$$\overline{P}_e = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\bar{\gamma}}{\pi}} \Gamma(1/2) = \frac{1}{2}\left[1 - \sqrt{\bar{\gamma}}\right] = \frac{1}{2}\left[1 - \sqrt{\frac{\gamma}{1+\gamma}}\right] \tag{11}$$

Problem 3.5.3 Solution

X is a Gaussian random variable with zero mean but unknown variance. We do know, however, that

$$P[|X| \leq 10] = 0.1 \quad (1)$$

We can find the variance Var$[X]$ by expanding the above probability in terms of the $\Phi(\cdot)$ function.

$$P[-10 \leq X \leq 10] = F_X(10) - F_X(-10) = 2\Phi\left(\frac{10}{\sigma_X}\right) - 1 \quad (2)$$

This implies $\Phi(10/\sigma_X) = 0.55$. Using Table 3.1 for the Gaussian CDF, we find that $10/\sigma_X = 0.15$ or $\sigma_X = 66.6$.

Problem 3.6.1 Solution

(a) Using the given CDF

$$P[X < -1] = F_X(-1^-) = 0 \quad (1)$$
$$P[X \leq -1] = F_X(-1) = -1/3 + 1/3 = 0 \quad (2)$$

Where $F_X(-1^-)$ denotes the limiting value of the CDF found by approaching -1 from the left. Likewise, $F_X(-1^+)$ is interpreted to be the value of the CDF found by approaching -1 from the right. We notice that these two probabilities are the same and therefore the probability that X is exactly -1 is zero.

(b)

$$P[X < 0] = F_X(0^-) = 1/3 \quad (3)$$
$$P[X \leq 0] = F_X(0) = 2/3 \quad (4)$$

Here we see that there is a discrete jump at $X = 0$. Approached from the left the CDF yields a value of 1/3 but approached from the right the value is 2/3. This means that there is a non-zero probability that $X = 0$, in fact that probability is the difference of the two values.

$$P[X = 0] = P[X \leq 0] - P[X < 0] = 2/3 - 1/3 = 1/3 \quad (5)$$

Problem 3.7.1 Solution

Since $0 \leq X \leq 1$, $Y = X^2$ satisfies $0 \leq Y \leq 1$. We can conclude that $F_Y(y) = 0$ for $y < 0$ and that $F_Y(y) = 1$ for $y \geq 1$. For $0 \leq y < 1$,

$$F_Y(y) = P\left[X^2 \leq y\right] = P\left[X \leq \sqrt{y}\right] \tag{1}$$

Since $f_X(x) = 1$ for $0 \leq x \leq 1$, we see that for $0 \leq y < 1$,

$$P\left[X \leq \sqrt{y}\right] = \int_0^{\sqrt{y}} dx = \sqrt{y} \tag{2}$$

Hence, the CDF of Y is

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ \sqrt{y} & 0 \leq y < 1 \\ 1 & y \geq 1 \end{cases} \tag{3}$$

By taking the derivative of the CDF, we obtain the PDF

$$f_Y(y) = \begin{cases} 1/(2\sqrt{y}) & 0 \leq y < 1 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$