Problem 4.1.1 Solution

(a) The probability $P[X \leq 2, Y \leq 3]$ can be found be evaluating the joint CDF $F_{X,Y}(x, y)$ at $x = 2$ and $y = 3$. This yields

$$P[X \leq 2, Y \leq 3] = F_{X,Y}(2, 3) = (1 - e^{-2})(1 - e^{-3}) \quad (1)$$

(b) To find the marginal CDF of X, $F_X(x)$, we simply evaluate the joint CDF at $y = \infty$.

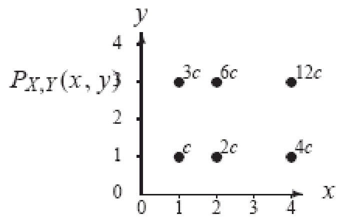
$$F_X(x) = F_{X,Y}(x, \infty) = \begin{cases} 1 - e^{-x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

(c) Likewise for the marginal CDF of Y, we evaluate the joint CDF at $X = \infty$.

$$F_Y(y) = F_{X,Y}(\infty, y) = \begin{cases} 1 - e^{-y} & y \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

Problem 4.2.1 Solution

In this problem, it is helpful to label points with nonzero probability on the X, Y plane:



(a) We must choose c so the PMF sums to one:

$$\sum_{x=1,2,4}\sum_{y=1,3} P_{X,Y}(x,y) = c\sum_{x=1,2,4} x \sum_{y=1,3} y = c[1(1+3)+2(1+3)+4(1+3)] = 28c \quad (1)$$

Thus $c = 1/28$.

(b) The event $\{Y < X\}$ has probability

$$P[Y < X] = \sum_{x=1,2,4}\sum_{y<x} P_{X,Y}(x,y) = \frac{1(0)+2(1)+4(1+3)}{28} = \frac{18}{28} \quad (2)$$

(c) The event $\{Y > X\}$ has probability

$$P[Y > X] = \sum_{x=1,2,4}\sum_{y>x} P_{X,Y}(x,y) = \frac{1(3)+2(3)+4(0)}{28} = \frac{9}{28} \quad (3)$$

(d) There are two ways to solve this part. The direct way is to calculate

$$P[Y = X] = \sum_{x=1,2,4}\sum_{y=x} P_{X,Y}(x,y) = \frac{1(1)+2(0)}{28} = \frac{1}{28} \quad (4)$$

The indirect way is to use the previous results and the observation that

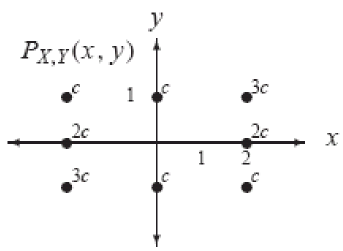
$$P[Y = X] = 1 - P[Y < X] - P[Y > X] = (1 - 18/28 - 9/28) = 1/28 \quad (5)$$

(e)

$$P[Y = 3] = \sum_{x=1,2,4} P_{X,Y}(x,3) = \frac{(1)(3)+(2)(3)+(4)(3)}{28} = \frac{21}{28} = \frac{3}{4} \quad (6)$$

Problem 4.2.2 Solution

On the X, Y plane, the joint PMF is



(a) To find c, we sum the PMF over all possible values of X and Y. We choose c so the sum equals one.

$$\sum_x \sum_y P_{X,Y}(x, y) = \sum_{x=-2,0,2} \sum_{y=-1,0,1} c|x + y| = 6c + 2c + 6c = 14c \tag{1}$$

Thus $c = 1/14$.

(b)

$$P[Y < X] = P_{X,Y}(0, -1) + P_{X,Y}(2, -1) + P_{X,Y}(2, 0) + P_{X,Y}(2, 1) \tag{2}$$
$$= c + c + 2c + 3c = 7c = 1/2 \tag{3}$$

(c)

$$P[Y > X] = P_{X,Y}(-2, -1) + P_{X,Y}(-2, 0) + P_{X,Y}(-2, 1) + P_{X,Y}(0, 1) \tag{4}$$
$$= 3c + 2c + c + c = 7c = 1/2 \tag{5}$$

(d) From the sketch of $P_{X,Y}(x, y)$ given above, $P[X = Y] = 0$.

(e)

$$P[X < 1] = P_{X,Y}(-2, -1) + P_{X,Y}(-2, 0) + P_{X,Y}(-2, 1) + P_{X,Y}(0, -1) + P_{X,Y}(0, 1) \tag{6}$$
$$= 8c = 8/14 \tag{7}$$

Problem 4.2.6 Solution

As the problem statement indicates, $Y = y < n$ if and only if

A: the first y tests are acceptable, and

B: test $y + 1$ is a rejection.

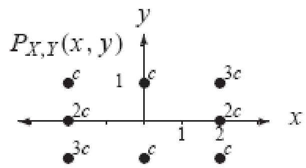
Thus $P[Y = y] = P[AB]$. Note that $Y \leq X$ since the number of acceptable tests before the first failure cannot exceed the number of acceptable circuits. Moreover, given the occurrence of AB, the event $X = x < n$ occurs if and only if there are $x - y$ acceptable circuits in the remaining $n - y - 1$ tests. Since events A, B and C depend on disjoint sets of tests, they are independent events. Thus, for $0 \leq y \leq x < n$,

$$
\begin{aligned}
P_{X,Y}(x, y) &= P[X = x, Y = y] && (1)\\
&= P[ABC] && (2)\\
&= P[A]P[B]P[C] && (3)\\
&= \underbrace{p^y}_{P[A]} \underbrace{(1-p)}_{P[B]} \underbrace{\binom{n-y-1}{x-y} p^{x-y}(1-p)^{n-y-1-(x-y)}}_{P[C]} && (4)\\
&= \binom{n-y-1}{x-y} p^x (1-p)^{n-x} && (5)
\end{aligned}
$$

Note that the remaining case, $y = x = n$ occurs when all n tests are acceptable and thus $P_{X,Y}(n, n) = p^n$.

Problem 4.3.2 Solution

On the X, Y plane, the joint PMF is



The PMF sums to one when $c = 1/14$.

(a) The marginal PMFs of X and Y are

$$P_X(x) = \sum_{y=-1,0,1} P_{X,Y}(x, y) = \begin{cases} 6/14 & x = -2, 2 \\ 2/14 & x = 0 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

$$P_Y(y) = \sum_{x=-2,0,2} P_{X,Y}(x, y) = \begin{cases} 5/14 & y = -1, 1 \\ 4/14 & y = 0 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

(b) The expected values of X and Y are

$$E[X] = \sum_{x=-2,0,2} x P_X(x) = -2(6/14) + 2(6/14) = 0 \quad (3)$$

$$E[Y] = \sum_{y=-1,0,1} y P_Y(y) = -1(5/14) + 1(5/14) = 0 \quad (4)$$

(c) Since X and Y both have zero mean, the variances are

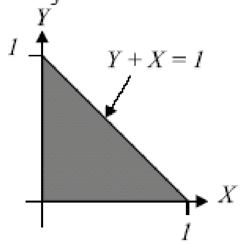
$$\text{Var}[X] = E[X^2] = \sum_{x=-2,0,2} x^2 P_X(x) = (-2)^2(6/14) + 2^2(6/14) = 24/7 \quad (5)$$

$$\text{Var}[Y] = E[Y^2] = \sum_{y=-1,0,1} y^2 P_Y(y) = (-1)^2(5/14) + 1^2(5/14) = 5/7 \quad (6)$$

The standard deviations are $\sigma_X = \sqrt{24/7}$ and $\sigma_Y = \sqrt{5/7}$.

Problem 4.4.1 Solution

(a) The joint PDF of X and Y is



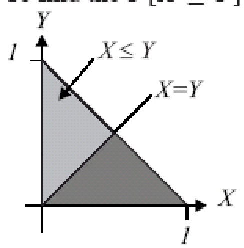
$$f_{X,Y}(x, y) = \begin{cases} c & x + y \leq 1, x, y \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

To find the constant c we integrate over the region shown. This gives

$$\int_0^1 \int_0^{1-x} c\, dy\, dx = cx - \frac{cx}{2}\Big|_0^1 = \frac{c}{2} = 1 \tag{2}$$

Therefore $c = 2$.

(b) To find the $P[X \leq Y]$ we look to integrate over the area indicated by the graph

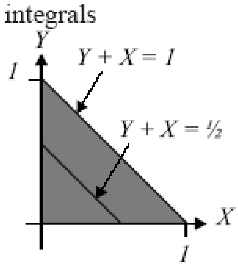


$$P[X \leq Y] = \int_0^{1/2} \int_x^{1-x} dy\, dx \tag{3}$$

$$= \int_0^{1/2} (2 - 4x)\, dx \tag{4}$$

$$= 1/2 \tag{5}$$

(c) The probability $P[X + Y \leq 1/2]$ can be seen in the figure. Here we can set up the following integrals



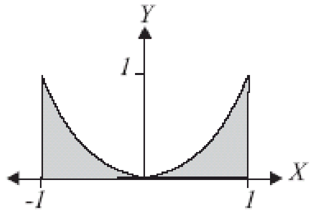
$$P[X + Y \leq 1/2] = \int_0^{1/2} \int_0^{1/2-x} 2\, dy\, dx \tag{6}$$

$$= \int_0^{1/2} (1 - 2x)\, dx \tag{7}$$

$$= 1/2 - 1/4 = 1/4 \tag{8}$$

Problem 4.5.4 Solution

The joint PDF of X and Y and the region of nonzero probability are



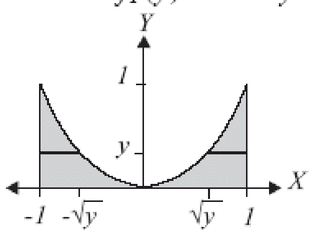
$$f_{X,Y}(x,y) = \begin{cases} 5x^2/2 & -1 \le x \le 1, 0 \le y \le x^2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

We can find the appropriate marginal PDFs by integrating the joint PDF.

(a) The marginal PDF of X is

$$f_X(x) = \int_0^{x^2} \frac{5x^2}{2}\, dy = \begin{cases} 5x^4/2 & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

(b) Note that $f_Y(y) = 0$ for $y > 1$ or $y < 0$. For $0 \le y \le 1$,



$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx \tag{3}$$

$$= \int_{-1}^{-\sqrt{y}} \frac{5x^2}{2}\, dx + \int_{\sqrt{y}}^{1} \frac{5x^2}{2}\, dx \tag{4}$$

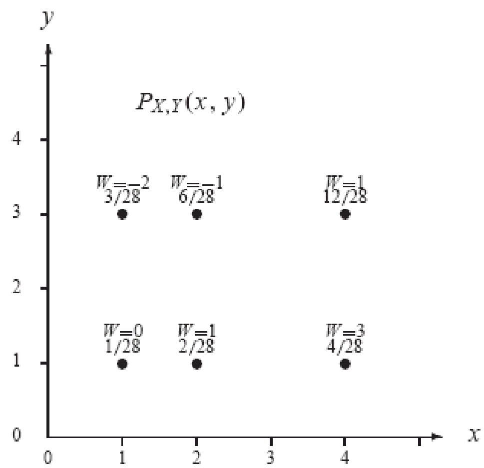
$$= 5(1 - y^{3/2})/3 \tag{5}$$

The complete expression for the marginal CDF of Y is

$$f_Y(y) = \begin{cases} 5(1 - y^{3/2})/3 & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Problem 4.6.1 Solution

In this problem, it is helpful to label points X, Y with nonzero probability along with the corresponding values of $W = X - Y$. From the statement of Problem 4.6.1, we have



(a) To find the PMF of W, we simply add the probabilities associated with each possible value of W:

$$P_W(-2) = P_{X,Y}(1, 3) = 3/28 \qquad P_W(-1) = P_{X,Y}(2, 3) = 6/28$$
$$P_W(0) = P_{X,Y}(1, 1) = 1/28 \qquad P_W(1) = P_{X,Y}(2, 1) + P_{X,Y}(4, 3) = 14/28$$
$$P_W(3) = P_{X,Y}(4, 1) = 4/28$$

For all other values of w, $P_W(w) = 0$.

(b) The expected value of W is

$$E[W] = \sum_w w P_W(w) = -2(3/28) + -1(6/28) + 0(1/28) + 1(14/28) + 3(4/28) = 1/2 \qquad (1)$$

(c)

$$P[W > 0] = P_W(1) + P_W(3) = 18/28 \qquad (2)$$

Problem 4.6.10 Solution

The position of the mobile phone is equally likely to be anywhere in the area of a circle with radius 16 km. Let X and Y denote the position of the mobile. Since we are given that the cell has a radius of 4 km, we will measure X and Y in kilometers. Assuming the base station is at the origin of the X, Y plane, the joint PDF of X and Y is

$$f_{X,Y}(x, y) = \begin{cases} \frac{1}{16\pi} & x^2 + y^2 \leq 16 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Since the radial distance of the mobile from the base station is $R = \sqrt{X^2 + Y^2}$, the CDF of R is

$$F_R(r) = P[R \leq r] = P[X^2 + Y^2 \leq r] \tag{2}$$

By changing to polar coordinates, we see that for $0 \leq r \leq 4$ km,

$$F_R(r) = \int_0^{2\pi} \int_0^r \frac{r'}{16\pi} \, dr' \, d\theta' = r^2/16 \tag{3}$$

So

$$F_R(r) = \begin{cases} 0 & r < 0 \\ r^2/16 & 0 \leq r < 4 \\ 1 & r \geq 4 \end{cases} \tag{4}$$

Then by taking the derivative with respect to r we arrive at the PDF

$$f_R(r) = \begin{cases} r/8 & 0 \leq r \leq 4 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

Problem 4.7.8 Solution

The joint PDF of X and Y is

$$f_{X,Y}(x,y) = \begin{cases} (x+y)/3 & 0 \le x \le 1, 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Before calculating moments, we first find the marginal PDFs of X and Y. For $0 \le x \le 1$,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_0^2 \frac{x+y}{3}\,dy = \frac{xy}{3} + \frac{y^2}{6}\Big|_{y=0}^{y=2} = \frac{2x+2}{3} \tag{2}$$

For $0 \le y \le 2$,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_0^1 (\frac{x}{3} + \frac{y}{3})\,dx = \frac{x^2}{6} + \frac{xy}{3}\Big|_{x=0}^{x=1} = \frac{2y+1}{6} \tag{3}$$

Complete expressions for the marginal PDFs are

$$f_X(x) = \begin{cases} \frac{2x+2}{3} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad f_Y(y) = \begin{cases} \frac{2y+1}{6} & 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

(a) The expected value of X is

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^1 x\frac{2x+2}{3}\,dx = \frac{2x^3}{9} + \frac{x^2}{3}\Big|_0^1 = \frac{5}{9} \tag{5}$$

The second moment of X is

$$E\left[X^2\right] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_0^1 x^2\frac{2x+2}{3}\,dx = \frac{x^4}{6} + \frac{2x^3}{9}\Big|_0^1 = \frac{7}{18} \tag{6}$$

The variance of X is $\text{Var}[X] = E[X^2] - (E[X])^2 = 7/18 - (5/9)^2 = 13/162$.

(b) The expected value of Y is

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\,dy = \int_0^2 y\frac{2y+1}{6}\,dy = \frac{y^2}{12} + \frac{y^3}{9}\Big|_0^2 = \frac{11}{9} \tag{7}$$

The second moment of Y is

$$E\left[Y^2\right] = \int_{-\infty}^{\infty} y^2 f_Y(y)\,dy = \int_0^2 y^2\frac{2y+1}{6}\,dy = \frac{y^3}{18} + \frac{y^4}{12}\Big|_0^2 = \frac{16}{9} \tag{8}$$

The variance of Y is $\text{Var}[Y] = E[Y^2] - (E[Y])^2 = 23/81$.

(c) The correlation of X and Y is

$$E[XY] = \iint xy f_{X,Y}(x, y)\, dx\, dy \tag{9}$$

$$= \int_0^1 \int_0^2 xy(\frac{x+y}{3})\, dy\, dx \tag{10}$$

$$= \int_0^1 (\frac{x^2y^2}{6} + \frac{xy^3}{9}\Big|_{y=0}^{y=2})\, dx \tag{11}$$

$$= \int_0^1 (\frac{2x^2}{3} + \frac{8x}{9})\, dx = \frac{2x^3}{9} + \frac{4x^2}{9}\Big|_0^1 = \frac{2}{3} \tag{12}$$

The covariance is $\text{Cov}[X, Y] = E[XY] - E[X]E[Y] = -1/81$.

(d) The expected value of X and Y is

$$E[X+Y] = E[X] + E[Y] = 5/9 + 11/9 = 16/9 \tag{13}$$

(e) By Theorem 4.15,

$$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y] + 2\,\text{Cov}[X, Y] = \frac{13}{162} + \frac{23}{81} - \frac{2}{81} = \frac{55}{162} \tag{14}$$

Problem 4.8.3 Solution

Given the event $A = \{X + Y \leq 1\}$, we wish to find $f_{X,Y|A}(x, y)$. First we find

$$P[A] = \int_0^1 \int_0^{1-x} 6e^{-(2x+3y)}\, dy\, dx = 1 - 3e^{-2} + 2e^{-3} \tag{1}$$

So then

$$f_{X,Y|A}(x, y) = \begin{cases} \frac{6e^{-(2x+3y)}}{1-3e^{-2}+2e^{-3}} & x + y \leq 1, x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Problem 4.9.13 Solution

The key to solving this problem is to find the joint PMF of M and N. Note that $N \geq M$. For $n > m$, the joint event $\{M = m, N = n\}$ has probability
begindmath0.3cm]

$$P[M = m, N = n] = P[\overbrace{dd\cdots d}^{\substack{m-1 \\ \text{calls}}}\, v\, \overbrace{dd\cdots d}^{\substack{n-m-1 \\ \text{calls}}}\, v] \tag{1}$$

$$= (1-p)^{m-1}p(1-p)^{n-m-1}p \tag{2}$$

$$= (1-p)^{n-2}p^2 \tag{3}$$

A complete expression for the joint PMF of M and N is

$$P_{M,N}(m,n) = \begin{cases} (1-p)^{n-2}p^2 & m = 1, 2, \ldots, n-1;\ n = m+1, m+2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

For $n = 2, 3, \ldots$, the marginal PMF of N satisfies

$$P_N(n) = \sum_{m=1}^{n-1}(1-p)^{n-2}p^2 = (n-1)(1-p)^{n-2}p^2 \tag{5}$$

Similarly, for $m = 1, 2, \ldots$, the marginal PMF of M satisfies

$$P_M(m) = \sum_{n=m+1}^{\infty}(1-p)^{n-2}p^2 \tag{6}$$

$$= p^2[(1-p)^{m-1} + (1-p)^m + \cdots] \tag{7}$$

$$= (1-p)^{m-1}p \tag{8}$$

The complete expressions for the marginal PMF's are

$$P_M(m) = \begin{cases} (1-p)^{m-1}p & m = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

$$P_N(n) = \begin{cases} (n-1)(1-p)^{n-2}p^2 & n = 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

Not surprisingly, if we view each voice call as a successful Bernoulli trial, M has a geometric PMF since it is the number of trials up to and including the first success. Also, N has a Pascal PMF since it is the number of trials required to see 2 successes. The conditional PMF's are now easy to find.

$$P_{N|M}(n|m) = \frac{P_{M,N}(m,n)}{P_M(m)} = \begin{cases} (1-p)^{n-m-1}p & n = m+1, m+2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

The interpretation of the conditional PMF of N given M is that given $M = m$, $N = m + N'$ where N' has a geometric PMF with mean $1/p$. The conditional PMF of M given N is

$$P_{M|N}(m|n) = \frac{P_{M,N}(m,n)}{P_N(n)} = \begin{cases} 1/(n-1) & m = 1, \ldots, n-1 \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

Given that call $N = n$ was the second voice call, the first voice call is equally likely to occur in any of the previous $n-1$ calls.