Problem 5.1.2 Solution

Whether a pizza has topping i is a Bernoulli trial with success probability $p_i = 2^{-i}$. Given that n pizzas were sold, the number of pizzas sold with topping i has the binomial PMF

$$P_{N_i}(n_i) = \begin{cases} \binom{n}{n_i} p_i^{n_i} (1 - p_i)^{n_i} & n_i = 0, 1, \ldots, n \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Since a pizza has topping i with probability p_i independent of whether any other topping is on the pizza, the number N_i of pizzas with topping i is independent of the number of pizzas with any other toppings. That is, $N_1, \ldots, N_4$ are mutually independent and have joint PMF

$$P_{N_1,\ldots,N_4}(n_1, \ldots, n_4) = P_{N_1}(n_1)\, P_{N_2}(n_2)\, P_{N_3}(n_3)\, P_{N_4}(n_4) \tag{2}$$

Problem 5.3.2 Solution

Since J_1, J_2 and J_3 are independent, we can write

$$P_{\mathbf{K}}(\mathbf{k}) = P_{J_1}(k_1)\, P_{J_2}(k_2 - k_1)\, P_{J_3}(k_3 - k_2) \tag{1}$$

Since $P_{J_i}(j) > 0$ only for integers $j > 0$, we have that $P_{\mathbf{K}}(\mathbf{k}) > 0$ only for $0 < k_1 < k_2 < k_3$; otherwise $P_{\mathbf{K}}(\mathbf{k}) = 0$. Finally, for $0 < k_1 < k_2 < k_3$,

$$P_{\mathbf{K}}(\mathbf{k}) = (1-p)^{k_1-1} p (1-p)^{k_2-k_1-1} p (1-p)^{k_3-k_2-1} p = (1-p)^{k_3-3} p^3 \tag{2}$$

Problem 5.3.8 Solution

In Problem 5.3.2, we found that the joint PMF of $\mathbf{K} = \begin{bmatrix} K_1 & K_2 & K_3 \end{bmatrix}'$ is

$$P_{\mathbf{K}}(\mathbf{k}) = \begin{cases} p^3(1-p)^{k_3-3} & k_1 < k_2 < k_3 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

In this problem, we generalize the result to n messages.

(a) For $k_1 < k_2 < \cdots < k_n$, the joint event

$$\{K_1 = k_1, K_2 = k_2, \cdots, K_n = k_n\} \tag{2}$$

occurs if and only if all of the following events occur

A_1	$k_1 - 1$ failures, followed by a successful transmission
A_2	$(k_2 - 1) - k_1$ failures followed by a successful transmission
A_3	$(k_3 - 1) - k_2$ failures followed by a successful transmission
$\vdots$	
A_n	$(k_n - 1) - k_{n-1}$ failures followed by a successful transmission

Note that the events $A_1, A_2, \ldots, A_n$ are independent and

$$P\left[A_j\right] = (1-p)^{k_j - k_{j-1} - 1} p. \tag{3}$$

Thus

$$\begin{aligned} P_{K_1,\ldots,K_n}(k_1, \ldots, k_n) &= P[A_1] P[A_2] \cdots P[A_n] & (4)\\ &= p^n (1-p)^{(k_1-1)+(k_2-k_1-1)+(k_3-k_2-1)+\cdots+(k_n-k_{n-1}-1)} & (5)\\ &= p^n (1-p)^{k_n - n} & (6) \end{aligned}$$

To clarify subsequent results, it is better to rename $\mathbf{K}$ as $\mathbf{K}_n$. That is, $\mathbf{K}_n = \begin{bmatrix} K_1 & K_2 & \cdots & K_n \end{bmatrix}'$, and we see that

$$P_{\mathbf{K}_n}(\mathbf{k}_n) = \begin{cases} p^n(1-p)^{k_n-n} & 1 \le k_1 < k_2 < \cdots < k_n, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

(b) For $j < n$,

$$P_{K_1,K_2,\ldots,K_j}\left(k_1, k_2, \ldots, k_j\right) = P_{\mathbf{K}_j}\left(\mathbf{k}_j\right). \tag{8}$$

Since $\mathbf{K}_j$ is just $\mathbf{K}_n$ with $n = j$, we have

$$P_{\mathbf{K}_j}\left(\mathbf{k}_j\right) = \begin{cases} p^j(1-p)^{k_j-j} & 1 \le k_1 < k_2 < \cdots < k_j, \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

(c) Rather than try to deduce $P_{K_i}(k_i)$ from the joint PMF $P_{\mathbf{K}_n}(\mathbf{k}_n)$, it is simpler to return to first principles. In particular, K_i is the number of trials up to and including the ith success and has the Pascal (i, p) PMF

$$P_{K_i}(k_i) = \binom{k_i - 1}{i - 1} p^i (1-p)^{k_i - i}. \tag{10}$$

Problem 5.5.4 Solution

Let X_i denote the finishing time of boat i. Since finishing times of all boats are iid Gaussian random variables with expected value 35 minutes and standard deviation 5 minutes, we know that each X_i has CDF

$$F_{X_i}(x) = P[X_i \le x] = P\left[\frac{X_i - 35}{5} \le \frac{x - 35}{5}\right] = \Phi\left(\frac{x - 35}{5}\right) \tag{1}$$

(a) The time of the winning boat is

$$W = \min(X_1, X_2, \ldots, X_{10}) \tag{2}$$

To find the probability that $W \le 25$, we will find the CDF $F_W(w)$ since this will also be useful for part (c).

$$F_W(w) = P[\min(X_1, X_2, \ldots, X_{10}) \le w] \tag{3}$$

$$= 1 - P[\min(X_1, X_2, \ldots, X_{10}) > w] \tag{4}$$

$$= 1 - P[X_1 > w, X_2 > w, \ldots, X_{10} > w] \tag{5}$$

Since the X_i are iid,

$$F_W(w) = 1 - \prod_{i=1}^{10} P[X_i > w] \tag{6}$$

$$= 1 - \left(1 - F_{X_i}(w)\right)^{10} \tag{7}$$

$$= 1 - \left(1 - \Phi\left(\frac{w - 35}{5}\right)\right)^{10} \tag{8}$$

Thus,

$$P[W \le 25] = F_W(25) = 1 - (1 - \Phi(-2))^{10}. \tag{9}$$

Since $\Phi(-2) = 1 - \Phi(2)$, we have that

$$P[W \le 25] = 1 - [\Phi(2)]^{10} = 0.2056 \tag{10}$$

(b) The finishing time of the last boat is $L = \max(X_1, \ldots, X_{10})$. The probability that the last boat finishes in more than 50 minutes is

$$P[L > 50] = 1 - P[L \le 50] \tag{11}$$

$$= 1 - P[X_1 \le 50, X_2 \le 50, \ldots, X_{10} \le 50] \tag{12}$$

Once again, we use the fact that the X_i are iid Gaussian (35, 5) random variables to write

$$P[L > 50] = 1 - \prod_{i=1}^{10} P[X_i \le 50] \tag{13}$$

$$= 1 - \left(F_{X_i}(50)\right)^{10} \tag{14}$$

$$= 1 - \left(\Phi\left(\frac{50 - 35}{5}\right)\right)^{10} \tag{15}$$

$$= 1 - (\Phi(3))^{10} = 0.0134 \tag{16}$$

(c) A boat will finish in negative time if and only iff the winning boat finishes in negative time, which has probability

$$F_W(0) = 1 - \left(1 - \Phi\left(\frac{0 - 35}{5}\right)\right)^{10} = 1 - (1 - \Phi(-7))^{10} = 1 - (\Phi(7))^{10} \tag{17}$$

Unfortunately, the table in the text has neother $\Phi(7)$ nor $Q(7)$. However, for those with access to MATLAB, or a programmable calculator, can find out that

$$Q(7) = 1 - \Phi(7) = 1.28 \times 10^{-12} \tag{18}$$

This implies that a boat finishes in negative time with probability

$$F_W(0) = 1 - (1 - 1.28 \times 10^{-12})^{10} = 1.28 \times 10^{-11}. \tag{19}$$

Problem 5.7.6 Solution

(a) From Theorem 5.13, $\mathbf{Y}$ has covariance matrix

$$\mathbf{C}_{\mathbf{Y}} = \mathbf{Q}\mathbf{C}_{\mathbf{X}}\mathbf{Q}' \tag{1}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{2}$$

$$= \begin{bmatrix} \sigma_1^2\cos^2\theta + \sigma_2^2\sin^2\theta & (\sigma_1^2 - \sigma_2^2)\sin\theta\cos\theta \\ (\sigma_1^2 - \sigma_2^2)\sin\theta\cos\theta & \sigma_1^2\sin^2\theta + \sigma_2^2\cos^2\theta \end{bmatrix}. \tag{3}$$

We conclude that Y_1 and Y_2 have covariance

$$\text{Cov}[Y_1, Y_2] = C_{\mathbf{Y}}(1,2) = (\sigma_1^2 - \sigma_2^2)\sin\theta\cos\theta. \tag{4}$$

Since Y_1 and Y_2 are jointly Gaussian, they are independent if and only if $\text{Cov}[Y_1, Y_2] = 0$. Thus, Y_1 and Y_2 are independent for all θ if and only if $\sigma_1^2 = \sigma_2^2$. In this case, when the joint PDF $f_{\mathbf{X}}(\mathbf{x})$ is symmetric in x_1 and x_2. In terms of polar coordinates, the PDF $f_{\mathbf{X}}(\mathbf{x}) = f_{X_1,X_2}(x_1, x_2)$ depends on $r = \sqrt{x_1^2 + x_2^2}$ but for a given r, is constant for all $\phi = \tan^{-1}(x_2/x_1)$. The transformation of $\mathbf{X}$ to $\mathbf{Y}$ is just a rotation of the coordinate system by θ preserves this circular symmetry.

(b) If $\sigma_2^2 > \sigma_1^2$, then Y_1 and Y_2 are independent if and only if $\sin\theta\cos\theta = 0$. This occurs in the following cases:

- $\theta = 0$: $Y_1 = X_1$ and $Y_2 = X_2$
- $\theta = \pi/2$: $Y_1 = -X_2$ and $Y_2 = -X_1$
- $\theta = \pi$: $Y_1 = -X_1$ and $Y_2 = -X_2$
- $\theta = -\pi/2$: $Y_1 = X_2$ and $Y_2 = X_1$

In all four cases, Y_1 and Y_2 are just relabeled versions, possibly with sign changes, of X_1 and X_2. In these cases, Y_1 and Y_2 are independent because X_1 and X_2 are independent. For other values of θ, each Y_i is a linear combination of both X_1 and X_2. This mixing results in correlation between Y_1 and Y_2.