Problem 6.1.4 Solution

We can solve this problem using Theorem 6.2 which says that

$$\text{Var}[W] = \text{Var}[X] + \text{Var}[Y] + 2\,\text{Cov}[X, Y] \tag{1}$$

The first two moments of X are

$$E[X] = \int_0^1 \int_0^{1-x} 2x\,dy\,dx = \int_0^1 2x(1-x)\,dx = 1/3 \tag{2}$$

$$E[X^2] = \int_0^1 \int_0^{1-x} 2x^2\,dy\,dx = \int_0^1 2x^2(1-x)\,dx = 1/6 \tag{3}$$

$$\tag{4}$$

Thus the variance of X is $\text{Var}[X] = E[X^2] - (E[X])^2 = 1/18$. By symmetry, it should be apparent that $E[Y] = E[X] = 1/3$ and $\text{Var}[Y] = \text{Var}[X] = 1/18$. To find the covariance, we first find the correlation

$$E[XY] = \int_0^1 \int_0^{1-x} 2xy\,dy\,dx = \int_0^1 x(1-x)^2\,dx = 1/12 \tag{5}$$

The covariance is

$$\text{Cov}[X, Y] = E[XY] - E[X]\,E[Y] = 1/12 - (1/3)^2 = -1/36 \tag{6}$$

Finally, the variance of the sum $W = X + Y$ is

$$\text{Var}[W] = \text{Var}[X] + \text{Var}[Y] - 2\,\text{Cov}[X, Y] = 2/18 - 2/36 = 1/18 \tag{7}$$

For this specific problem, it's arguable whether it would easier to find $\text{Var}[W]$ by first deriving the CDF and PDF of W. In particular, for $0 \le w \le 1$,

$$F_W(w) = P[X + Y \le w] = \int_0^w \int_0^{w-x} 2\,dy\,dx = \int_0^w 2(w-x)\,dx = w^2 \tag{8}$$

Hence, by taking the derivative of the CDF, the PDF of W is

$$f_W(w) = \begin{cases} 2w & 0 \le w \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

From the PDF, the first and second moments of W are

$$E[W] = \int_0^1 2w^2\,dw = 2/3 \qquad E\left[W^2\right] = \int_0^1 2w^3\,dw = 1/2 \tag{10}$$

The variance of W is $\text{Var}[W] = E[W^2] - (E[W])^2 = 1/18$. Not surprisingly, we get the same answer both ways.

Problem 6.6.1 Solution

We know that the waiting time, W is uniformly distributed on [0,10] and therefore has the following PDF.

$$f_W(w) = \begin{cases} 1/10 & 0 \leq w \leq 10 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

We also know that the total time is 3 milliseconds plus the waiting time, that is $X = W + 3$.

(a) The expected value of X is $E[X] = E[W + 3] = E[W] + 3 = 5 + 3 = 8$.

(b) The variance of X is $\text{Var}[X] = \text{Var}[W + 3] = \text{Var}[W] = 25/3$.

(c) The expected value of A is $E[A] = 12E[X] = 96$.

(d) The standard deviation of A is $\sigma_A = \sqrt{\text{Var}[A]} = \sqrt{12(25/3)} = 10$.

(e) $P[A > 116] = 1 - \Phi(\frac{116-96}{10}) = 1 - \Phi(2) = 0.02275$.

(f) $P[A < 86] = \Phi(\frac{86-96}{10}) = \Phi(-1) = 1 - \Phi(1) = 0.1587$

Problem 6.6.2 Solution

Knowing that the probability that voice call occurs is 0.8 and the probability that a data call occurs is 0.2 we can define the random variable D_i as the number of data calls in a single telephone call. It is obvious that for any i there are only two possible values for D_i, namely 0 and 1. Furthermore for all i the D_i's are independent and identically distributed withe the following PMF.

$$P_D(d) = \begin{cases} 0.8 & d = 0 \\ 0.2 & d = 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

From the above we can determine that

$$E[D] = 0.2 \qquad \text{Var}[D] = 0.2 - 0.04 = 0.16 \tag{2}$$

With the previous descriptions, we can answer the following questions.