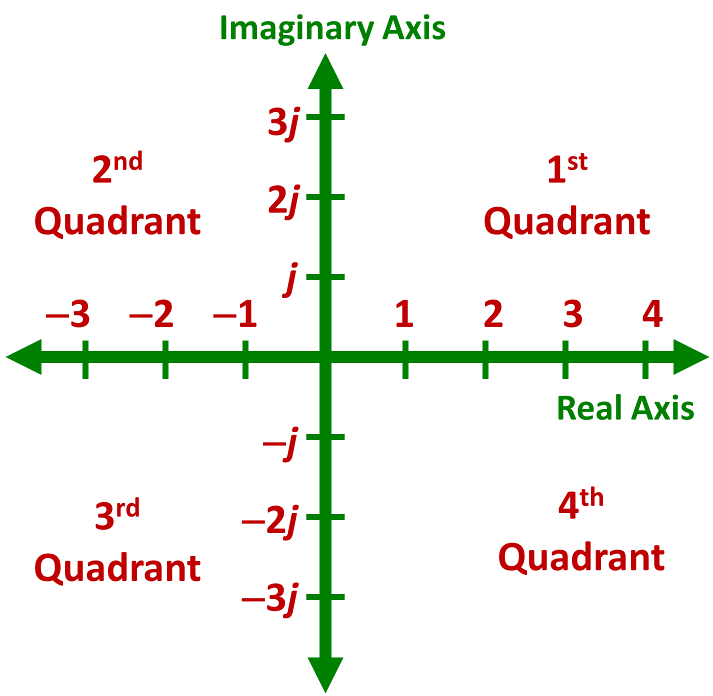
AC RL and RC Circuits

- **When a sinusoidal AC voltage is applied to an RL or RC circuit, the relationship between voltage and current is altered.**
- **The voltage and current still have the same frequency and cosine-wave shape, but voltage and current no longer rise and fall together.**
- **To solve for currents in AC RL/RC circuits, we need some additional mathematical tools:**
  - **Using the complex plane in problem solutions.**
  - **Using transforms to solve for AC sinusoidal currents.**

Imaginary Numbers

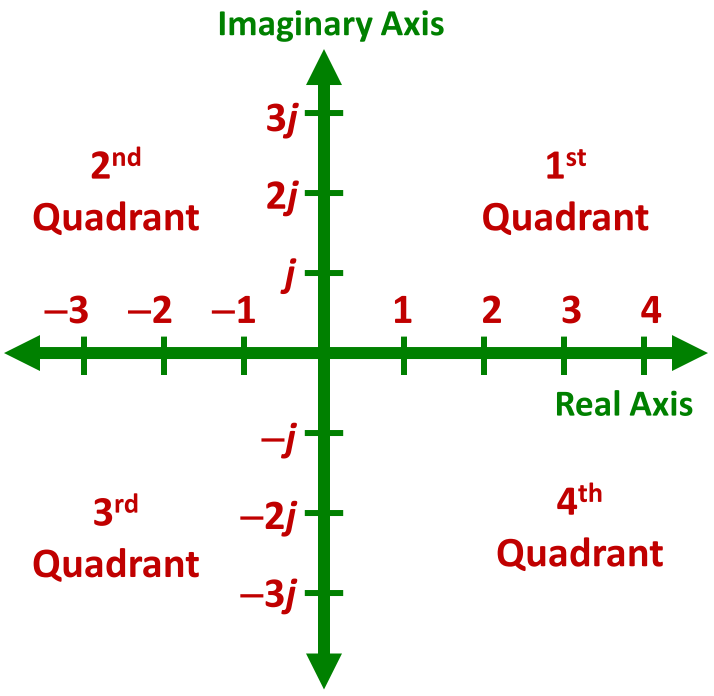
- Solutions to science and engineering problems often involve $\sqrt{-1}$.
- Scientists define $i = -\sqrt{-1}$.
- As we EE's use *i* for AC current, we define $j = +\sqrt{-1}$.
- Thus technically, $j = -i$, but that does not affect the math.
- Solutions that involve *j* are said to use "imaginary numbers."
- Imaginary numbers can be envisioned as existing with real numbers in a two-dimensional plane called the "Complex Plane."



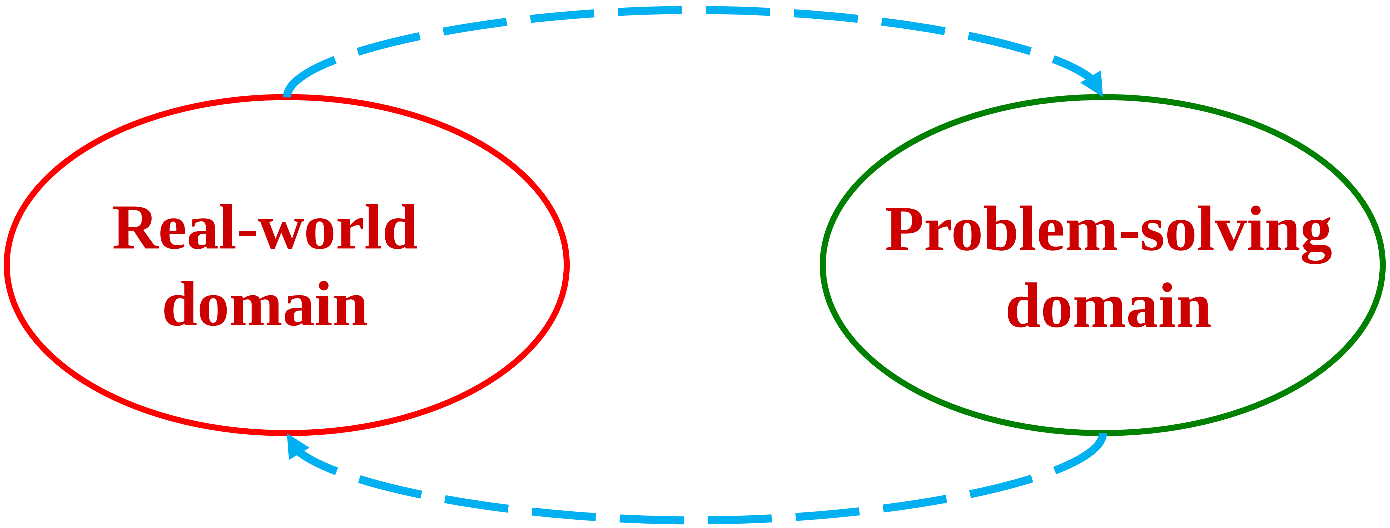
The Complex Plane

The Complex Plane

- **In the complex plane, imaginary numbers lie on the y-axis, real numbers on the x-axis, and complex numbers (mixed real and imaginary) lie off-axis.**
- **For example, 4 is on the +x axis, –8 is on the –x axis, *j*6 is on the + y axis, and –*j*14 is on the –y axis.**
- **Complex numbers like 6+*j*4, or –12 –*j*3 lie off-axis, the first in the first quadrant, and the second in the third quadrant.**



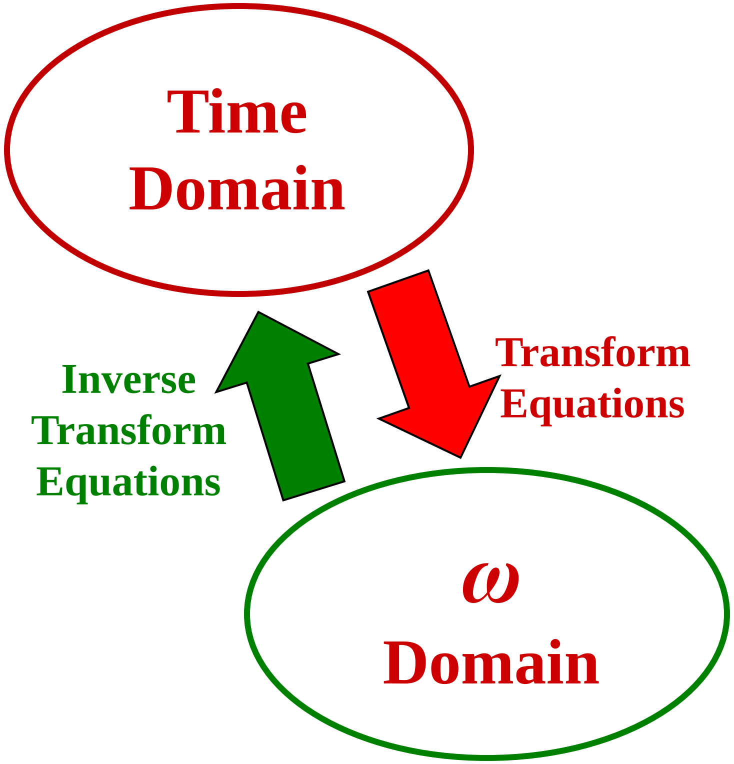
Why Transforms?



- Transforms move a problem from the real-world domain, where it is hard to solve, to an alternate domain where the solution is easier.
- Sinusoidal AC problems involving *R-L-C* circuits are hard to solve in the "real" time domain but easier to solve in the ω-domain.

The ω Domain

- **In the time domain, *RLC* circuit problems must be solved using calculus.**
- **However, by transforming them to the ω domain (a radian frequency domain, $\omega = 2\pi f$), the problems become algebra problems.**
- **A catch: We need transforms to get the problem to the ω domain, and inverse transforms to get the solutions back to the time domain!**



A Review of Euler's Formula

- **You should remember Euler's formula from trigonometry (if not, get out your old trig textbook and review):** $e^{\pm jx} = \cos x \pm j \sin x$ **.**
- **The alternate expression for $e^{\pm jx}$ is a <u>complex number</u>. The real part is cos x and the imaginary part is ± $j$sin x.**
- **We can say that cos x = Re{$e^{\pm jx}$} and ±$j$sin x =Im{$e^{\pm jx}$}, where Re = "real part" and Im = "imaginary part."**
- **We usually express AC voltage as a <u>cosine function</u>. That is, an AC voltage $v(t)$ and be expressed as $v(t) = V_p \cos \omega t$, where V_p is the peak AC voltage.**
- **Therefore we can say that $v(t) = V_p \cos \omega t = V_p \mathrm{Re}\{e^{\pm j\omega t}\}$. This relation is important in developing inverse transforms.**

Transforms into the ω Domain

Element	Time Domain	ω Domain Transform
AC Voltage	$V_p \cos \omega t$	V_p
Resistance	R	R
Inductance	L	$j\omega L$
Capacitance	C	$1/j\omega C$

- **The time-domain, sinusoidal AC voltage is normally represented as a cosine function, as shown above.**
- ***R*, *L* and *C* are in Ohms, Henrys and Farads.**
- **Skipping some long derivations (which you will get in EE 3301), transforms for the ω domain are shown above.**
- **Notice that the AC voltage ω-transform has no frequency information. However, frequency information is carried in the *L* and *C* transforms.**

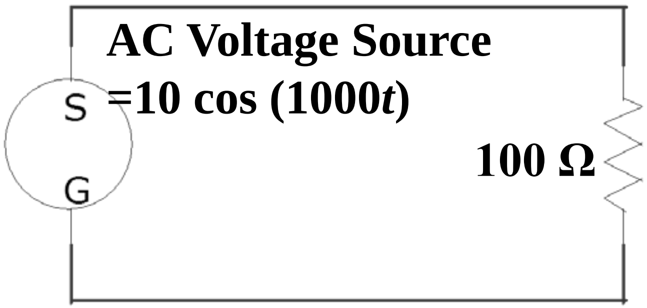
Comments on ω Transforms

Element	Time Domain	ω Domain Transform
AC Voltage	$V_p \cos \omega t$	V_p
Resistance	R	R
Inductance	L	$j\omega L$
Capacitance	C	$1/j\omega C$

- **Because we are studying constant-frequency sinusoidal AC circuits, the *ω*-domain transforms are constants.**
- **This is a considerable advantage over the time-domain situation, where *t* varies constantly (which is why solving for sinusoidal currents in the time domain is a calculus problem).**
- **Two other items:**
  - **In the *ω*-domain, the units of *R*, *jωL*, and *1/jωC* are Ohms.**
  - **In the *ω*-domain, Ohm's Law and Kirchoff's voltage and current laws still hold.**

Solving for Currents in the ω Domain

- **Solving problems in the frequency domain:**
  - **Given a circuit with the AC voltage shown, and only a resistor in the circuit, then the transform of the voltage is 10. *R* transforms directly as 100.**
  - **Solving for the circuit current, *I*=*V*/*R*, or *I*=10/100 = 0.1 A.**
  - **This current is the *ω*-domain answer. It must be inverse-transformed to the time domain to obtain a usable answer.**

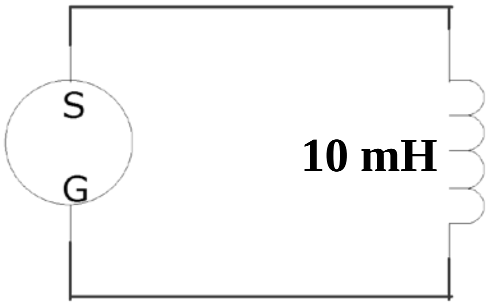


ω-domain voltage = V_p = 10

ω-domain current = V_p/R = 10/100 = 0.1 ampere

An ω Domain Solution for an *L* Circuit

- The ω-domain voltage is still 10.
- The ω-domain transform of $L = j\omega L = j(1000)10(10)^{-3}. = j10$.
- The units of the *L* transform is in Ohms (Ω), i.e., the ω-domain transform of *L* is $j10\ \Omega$.
- The value ωL is called **inductive reactance (*X*)**. The quantity $j\omega L$ is called **impedance (*Z*)**.
- Finding the current: $I = V/Z = 10/j10 = 1/j = -j$ (rationalizing).
- Time-domain answer in a few slides!



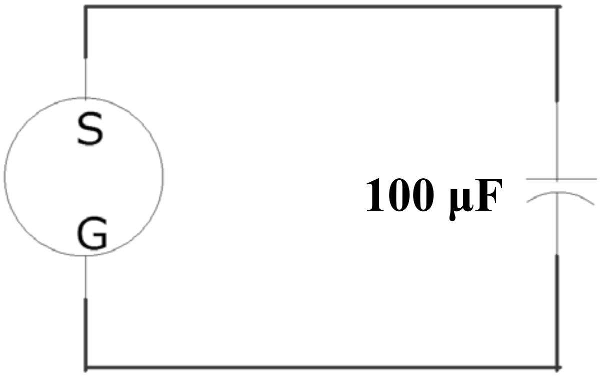
AC Voltage
=10 cos (1000*t*)

ω-domain voltage = $V_p = 10$

ω-domain current = $V_p / j\omega L$
$= 10/j10 = -j1 = -j$ ampere

An ω Domain Solution for a *C* Circuit

- The ω-domain voltage still = 10.
- The ω-domain transform of $C = 1/j\omega C$ $=1/j(1000)100(10)^{-6}. = 1/j0.1 = -j10$.
- The units of the *C* transform is in Ohms (Ω), i.e., the ω-domain transform of *C* is $-j10\ \Omega$.
- The value $1/\omega C$ is called capacitive reactance, and $1/j\omega C$ is also called impedance (here, capacitive impedance).
- Finding the current: $I = V/Z = 10/-j10 = 1/-j = j1$ (rationalizing) $= j$.
- Time-domain answer coming up!



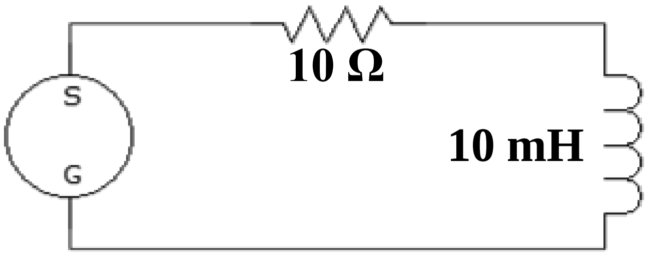
AC Voltage
=10 cos (1000t)

ω-domain voltage $= V_p = 10$

ω-domain current $= V_p/(1/j\omega C)$
$= 10/-j10 = j1 = j$ amperes

An *RL* ω-Domain Solution

- The ω-domain voltage still = 10.
- The ω-domain impedance is $10+j10$.
- Resistance is still called resistance in the ω-domain. The *R* and *L* transforms are called impedance, and a combination of resistance and imaginary impedances is also called impedance.
- Note: all series impedances add directly in the ω-domain.
- Finding the current: $I = V/Z = 10/(10+j10) =$ (rationalizing) $(100–j100)/200 = 0.5–j0.5$.
- Time-domain answers next!



AC Voltage =10 cos (1000t)

ω-domain voltage = $V_p = 10$

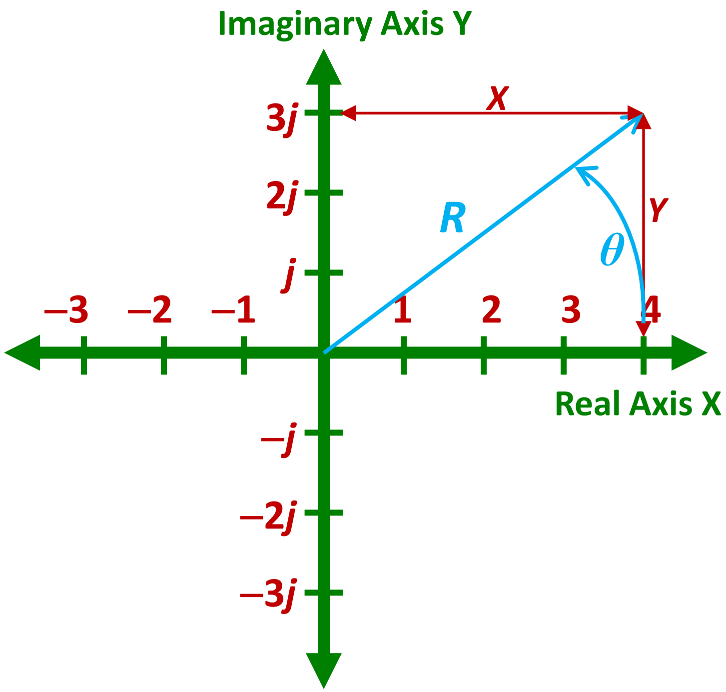
ω-domain current = $V_p /(R+j\omega L) = 10/(10+j10) = 0.5–j0.5$ ampere

Inverse Transforms

- **Our ω-domain solutions do us no good, since we are inhabitants of the time domain.**
- **We required a methodology for inverse transforms, mathematical expressions that can convert the frequency domain currents we have produced into their time-domain counterparts.**
- **It turns out that there is a fairly straightforward inverse transform methodology which we can employ.**
- **First, some preliminary considerations.**

Cartesian-to-Polar Transformations

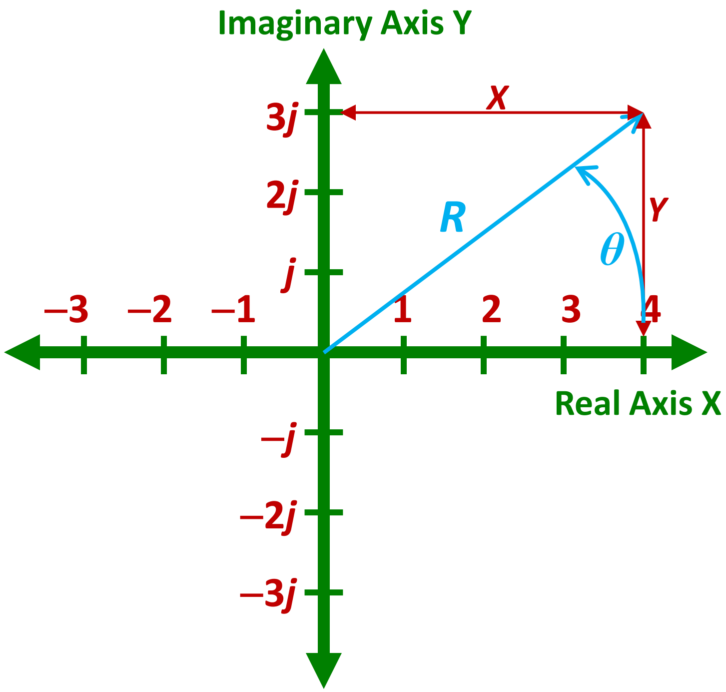
- **Our *ω*-domain answers are complex numbers – currents expressed in the *X-Y* coordinates of the complex plane.**
- **Coordinates in a two-dimensional plane may also be expressed in <u>*R-θ* coordinates</u>: a radius length *R* plus a counterclockwise angle *θ* from the positive X-axis (at right).**
- **That is, <u>there is a coordinate *R*,*θ* that can express an equivalent position to an *X*,*Y* coordinate</u>.**



Cartesian-to-Polar Transformations (2)

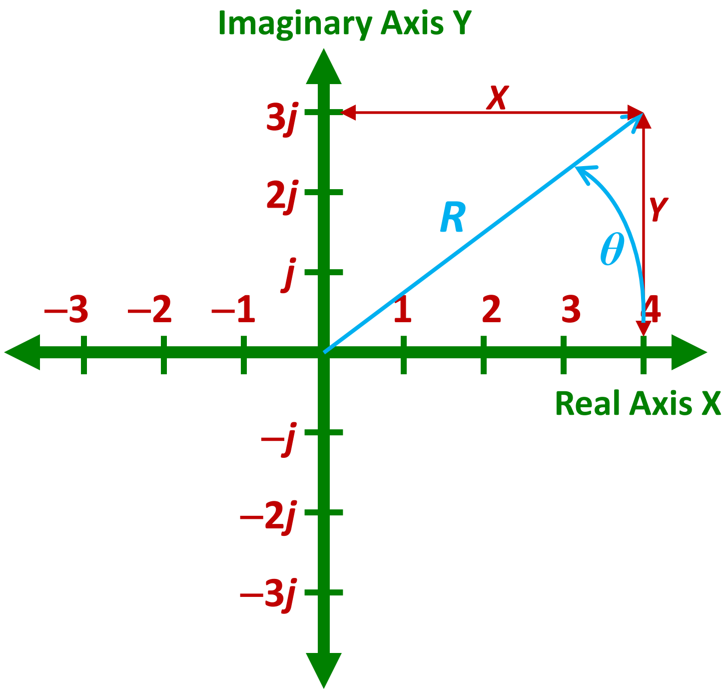
- The R,θ coordinate is equivalent to the X,Y coordinate if $\theta = \arctan(Y/X)$ and $R = \sqrt{X^2 + Y^2}$.
- In our X-Y plane, the X axis is the real axis, and the Y axis is the imaginary axis. Thus the coordinates of a point in the complex plane with (for example) X coordinate A and Y coordinate +B is A+*j*B.
- Now, remember Euler's formula:

$$e^{\pm jx} = \cos x \pm j \sin x$$



Cartesian-to-Polar Transformations (3)

- If $e^{\pm jx} = \cos x \pm j \sin x$, then $Re^{\pm jx} = R \cos x \pm Rj \sin x$.
- But in our figure, $R\cos\theta = X$, and $R\sin\theta = Y$.
- Or, $Re^{\pm j\theta} = X \pm jY$!
- What this says is that when we convert our ω-domain AC current answers into polar coordinates, we can express the values in $Re^{\pm j\theta}$ format as well as R,θ format.
- The $Re^{\pm j\theta}$ is very important in the inverse transforms.



Inverse Transform Methodology

- **We seek a time-domain current solution of the form** $i(t) = I_p \cos(\omega t)$**. where** I_p **is some peak current.**
- **This is difficult to do with the** ω**-domain answer in Cartesian (A±*j*B) form.**
- **So, we convert the** ω**-domain current solution to** R,θ **format, then convert that form to the** $Re^{\pm j\theta}$ **form, where we know that** $\theta = \arctan(Y/X)$**, and** $R = \sqrt{X^2 + Y^2}$**.**
- **Once the** ω**-domain current is in** $Re^{\pm j\theta}$ **form (and skipping a lot of derivation), we can get the time-domain current as follows:**

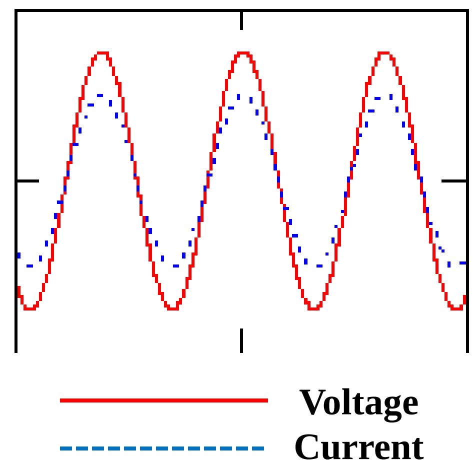
Inverse Transform Methodology (2)

- **Given the $Re^{\pm j\theta}$ expression of the ω-domain current, we have only to do two things:**
  - **Multiply the $Re^{\pm j\theta}$ expression by $e^{j\omega t}$.**
  - **Take the real part.**
- **This may seem a little magical at this point, but remember, Re ($e^{j\omega t}$) is cos ωt, and we are looking for a current that is a cosine function of time.**
- **We can see examples of this methodology by converting our four ω-domain current solutions to real time-domain answers.**

Transforming Solutions

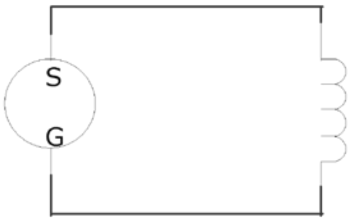
- In the resistor case, our *ω*-domain current is a real number, 0.1 A. Then *X*=0.1, *Y*=0.
- Then $R=\sqrt{X^2+Y^2}=\sqrt{(0.1)^2}=0.1$, and θ = arctan*Y/X* = arctan 0 = 0.
- Thus current = Re $\{0.1\, e^{j\omega t}e^{j0}\}$ = 0.1Re $\{0.1\, e^{j\omega t}\}$ = 0.1 cos1000*t* A.
- Physically, this means that the AC current is cosinusoidal, like the voltage. It rises and falls in lock step with the voltages, and has a maximum value of 0.1 A (figure at right).

ω-domain current = V_p/R = 10/100 = 0.1 ampere

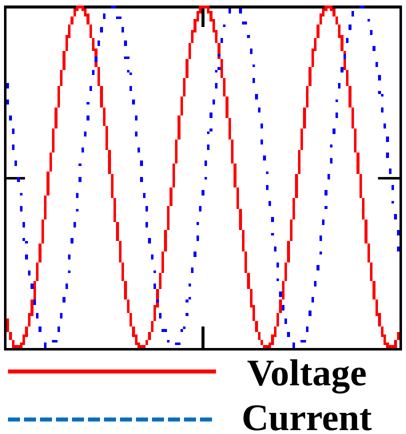


Transforming Solutions (2)

- For the inductor circuit, $I = -j1 = -j$.
- Converting to polar: $R = \sqrt{X^2 + Y^2} = \sqrt{(1)^2} = 1$
- θ = arctanY/X = arctan –1/0 = arctan – ∞ = –90°.
- I_ω = 1, –90° = $1e^{-j90°} = e^{-j90°}$.
- Multiplying by $e^{j\omega t}$ and taking the real part: $i(t)$ = Re{$e^{j\omega t} \cdot e^{-j90°}$} = Re{$e^{j(\omega t - 90°)}$} = (1)cos($\omega t$–90°) = cos($\omega t$–90°) A.
- Physical interpretation: $i(t)$ is a maximum of 1 A, is cosinusoidal like the voltage, but lags the voltage by exactly 90° (plot at right).
- The angle θ between voltage and current is called the phase angle. Cos θ is called the power factor, a measure of power dissipation in an inductor or capacitor circuit.



ω-domain current = $V_p / j\omega L$ = $10/j10 = -j1 = -j$ ampere

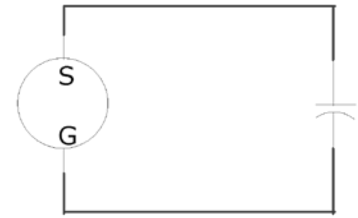


Transforming Solutions (3)

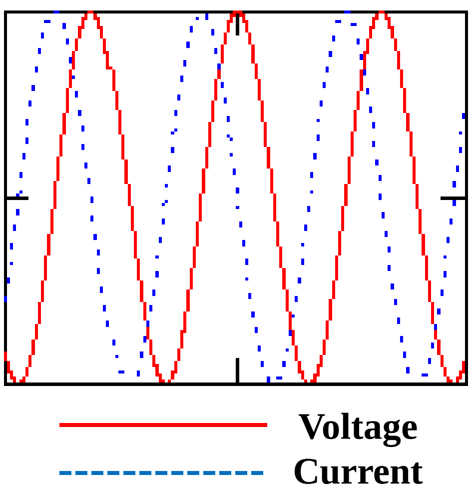
- **For the capacitor circuit, $I = -j$ A.**
- **Converting to polar:**

$$R = \sqrt{X^2 + Y^2} = \sqrt{(-1)^2} = 1 \quad .$$

- **θ = arctan Y/X = arctan 1/0 = arctan ∞ = 90°, so that I_ω = 1, 90° = $e^{j90°}$.**
- **Multiplying by $e^{j\omega t}$ and taking the real part: $i(t) = \text{Re}\{e^{j\omega t} \cdot e^{j90°}\} = \text{Re}\{e^{j(\omega t + 90°)}\}$ = cos(ωt+90°) A.**
- **Physically, *i(t)* has a maximum amplitude of 1 A, is cosinusoidal like the voltage, but <u>leads</u> the voltage <u>by exactly 90°</u> (figure at right).**

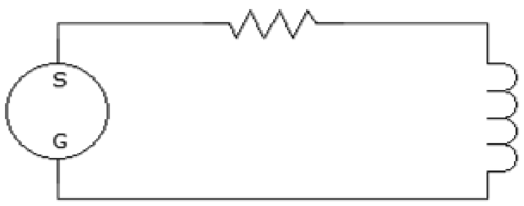


ω-domain current = $V_p/(1/j\omega C)$ = 1/–j0.1 = j10 amperes



Transforming Solutions (4)

- For the *RL* circuit, *I* = 0.5–*j*0.5 ampere.
- Converting to polar:
  $$R = \sqrt{X^2 + Y^2} = \sqrt{(0.5)^2 + (-0.5)^2} \approx 0.707.$$
- And θ = arctan*Y/X* = arctan –0.5/0.5 = arctan –1= –45°; I_ω = 0.707, –90° = $0.707e^{-j45°}$.
- Multiplying by $e^{j\omega t}$ and taking the real part: $i(t) = \mathrm{Re}\{0.707e^{j\omega t} \cdot e^{-j45°}\} = 0.707\mathrm{Re}\{e^{j(\omega t - 45°)}\}$ = $0.707\cos(\omega t - 45°) = 0.707\cos(\omega t - 45°)$ A.
- Note the physical interpretation: *i(t)* has a maximum amplitude of 0.707 A, is cosinusoidal like the voltage, and lags the voltage by 45°. Lagging current is an inductive characteristic, but it is less than 90°, due to the influence of the resistor.



ω-domain current = $V_p/(R+j\omega L)$ = 10/(10+*j*10) = 0.5–*j*0.5 ampere

Summary: Solving for Currents Using ω Transforms

- **Transform values to the ω-domain:**

Element	Time Domain	ω Domain Transform
AC Voltage	$V_p \cos \omega t$	V_p
Resistance	R	R
Inductance	L	$j\omega L$
Capacitance	C	$1/j\omega C$

- **Solve for I_ω, using Ohm's and Kirchoff's laws.**
  - **Solution will be of the form A±*j*B (Cartesian complex plane).**
- **Use inverse transforms to obtain *i(t)*.**
  - **Convert the Cartesian solution (A±*j*B) to *R*,*θ* format and thence to $Re^{\pm j\theta}$ form.**
  - **Multiply by $e^{\pm j\omega t}$ and take the real part to get a cosine-expression for *i(t)*.**

Measuring AC Current Indirectly

- **Because we do not have current probes for the oscilloscope, we will use an indirect measurement to find *i(t)* (reference Figs. 11 and 13 in Exercise 5).**
- **As the circuit resistance is real, it does not contribute to the phase angle of the current. Then a measure of voltage across the circuit resistance is a direct measure of the phase of *i(t)*.**
- **Further, a measure of the Δt between the *i,v* peaks is a direct measure of the phase difference in seconds.**
- **We will use this method to determine the actual phase angle and magnitude of the current in Lab. 5.**

Discovery Exercises

- **Lab. 5 includes two exercises that uses inductive and capacitive impedance calculations to allow the discovery of the equivalent inductance of series inductors and the equivalent capacitance of series capacitors.**
- **Question 7.6 then asks you to infer the equivalent inductance of parallel inductors and the equivalent capacitance of parallel capacitors.**
- **Although you are really making an educated guess at that point, you can validate your guess using ω-domain circuit theory, with one additional bit of knowledge not covered in the lab text:**
  - **In the ω-domain, parallel impedances add reciprocally, just like resistances in a DC circuit.**
  - **(Remember that in the ω-domain, series impedances add directly).**