Sub-Convergence with Multiple Factors

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Erasmo Papagni asked a question regarding the relationship between multivariate factors and the relative convergence. Before answering the question, we study underlying econometric models for the relative convergence. Assume that the variable of interest, \( y_{it} \), has a single common factor.

\[
y_{it} = a_i + \lambda_{1i} F_{1t} + y_{it}^o,
\]

where \( y_{it}^o \) is a pure idiosyncratic component. Further assume that \( F_{1t} \) has a stochastic trend. That is, \( F_{1t} = c_0 + c_1 t + u_{1t} \) with \( u_{1t} = u_{1t-1} + \varepsilon_{1t} \) where \( \varepsilon_{1t} \) is a stationary variable. Next, observe this.

\[
y_{it} = \left( \lambda_{1i} + \frac{a_i + y_{it}^o}{F_{1t}} \right) F_{1t} = \lambda_{1i,t} F_{1t}. \tag{2}
\]

Phillips and Sul (2007) modeled \( y_{it} \) as a product between a time varying factor loading \( \lambda_{1i,t} \) and a common factor.

Consider the case with two factors:

\[
y_{it} = a_i + \lambda_{1i,t} F_{1t} + \lambda_{2i,t} F_{2t} + y_{it}^o = \left( \frac{a_i + \lambda_{1i} F_{1t} + \lambda_{2i} F_{2t} + y_{it}^o}{F_t} \right) F_t = b_{it} F_t, \tag{3}
\]

where \( F_t \) can be either \( F_{1t} \) or \( F_{2t} \). It does not matter at all how to rewrite a model.

Next, consider (1) again to evaluate the meaning of the relative convergence. If \( \lambda_{1i} = \lambda_1 \) for all \( i \), then the relative convergence holds since

\[
\lambda_{1i,t} = \lambda_1 + \frac{a_i + y_{it}^o}{F_{1t}} \to \lambda_1 \text{ as } t \to \infty.
\]
Note that $a_i/F_{1t} = a_i / (c_0 + c_1 t + u_{1t}) \to 0$ as $t \to \infty$. Similarly, $y_{it} \to (c_0 + c_1 t + u_{1t}) \to 0$ as well. Suppose that
\[
\lambda_{1i} = \begin{cases} 
\lambda_{11} & \text{if } i \in G_1 \\
\lambda_{12} & \text{if } i \in G_2 
\end{cases},
\]
then as $t \to \infty$,
\[
y_{it} \to \begin{cases} 
\lambda_{11} F_{1t} & \text{if } i \in G_1 \\
\lambda_{12} F_{1t} & \text{if } i \in G_2 
\end{cases}.
\]
In other words, $y_{it}$ has two sub-convergent clubs. The cross-sectional averages of $y_{it}$ in $G_1$ and $G_2$ with a very large $n$ must look like this.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1}
\caption{Estimated Common Factors in Sub-Convergent Panels}
\end{figure}

It is because the cross-sectional average becomes a good proxy for the common factor for each sub-convergent panel. As $n \to \infty$, the average approaches to its true value. Nonetheless, the gap between two factors must be increasing over time if $F_t$ includes a trend component. Furthermore, the general pattern must be very similar.

With multiple factors, one may have various different cases. For example, assume that $\lambda_{1i} = \lambda_1$ for all $i$ & $\lambda_{2i} = \lambda_2$ in (3). Further we assume that
\[
y_{it} = b_{it} F_t = \begin{cases} 
\lambda_{11} F_{1t} + y_{it}^0 & \text{if } i \in G_1 \\
\lambda_{21} F_{2t} + y_{it}^0 & \text{if } i \in G_2 
\end{cases}.
\]
Then as $t \to \infty$, we have
\[
y_{it} \to \begin{cases} 
\lambda_{11} F_{1t} & \text{if } i \in G_1 \\
\lambda_{21} F_{2t} & \text{if } i \in G_2 
\end{cases}.
\]
In this case, the time varying pattern of the cross-sectional average in each sub-convergent club must be different.

Nonetheless, the number of sub-convergent clubs is not necessarily equal to the number of common factors. With a single factor case in (1), we already showed that there are possibly many sub-convergent clubs. With two factors, there must be more than two sub-convergent clubs (as $n$ and $t \to \infty$).