1. A basic arithmetic expression is composed of characters from the set \{1, +, ×\} and parenthesis. Almost every non-negative integer can be represented by more than one basic arithmetic expression. For example, all of the following basic arithmetic expressions represent the integer 14:

\[
1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1
\]

\[
((1 + 1) \times (1 + 1 + 1 + 1)) + ((1 + 1) \times (1 + 1))
\]

\[
(1 + 1) \times (1 + 1 + 1 + 1 + 1 + 1 + 1)
\]

\[
(1 + 1) \times ((1 + 1 + 1) \times (1 + 1)) + 1
\]

Describe and analyze an algorithm to compute, given an integer \(n\) as input, the minimum number of 1s in a basic arithmetic expression whose value is equal to \(n\). The number of parentheses doesn’t matter, just the number of 1s. For example, when \(n = 14\), your algorithm should return 8 for the final expression above. The running time of your algorithm should be bounded by a small polynomial function of \(n\).

2. Let \(T\) be a rooted binary tree with \(n\) vertices, and let \(k \leq n\) be a positive integer. We would like to mark \(k\) vertices in \(T\) so that every vertex has a nearby marked ancestor. More formally, we define the clustering cost of any subset \(K\) of vertices as

\[
\text{cost}(K) = \max_v \text{cost}(v, K),
\]

where the maximum is taken over all vertices \(v\) in the tree, and \(\text{cost}(v, K)\) is the distance from \(v\) to its nearest ancestor in \(K\):

\[
\text{cost}(v, K) = \begin{cases} 
0 & \text{if } v \in K \\
\infty & \text{if } v \text{ is the root of } T \text{ and } v \notin K \\
1 + \text{cost}(\text{parent}(v)) & \text{otherwise}
\end{cases}
\]

In particular, \(\text{cost}(K) = \infty\) if \(K\) excludes the root of \(T\).

(a) Describe a dynamic programming algorithm to compute, given the tree \(T\) and an integer \(r\), the size of the smallest subset of vertices whose clustering cost is at most \(r\). For full credit, your algorithm should run in \(O(nr)\) time.
Figure 1: A subset of six vertices in a binary tree with clustering cost 3. The other vertices are labeled with the distance to their nearest ancestor in the subset.

(b) Describe an algorithm to compute, given the tree $T$ and an integer $k$, the minimum clustering cost of any subset of $k$ vertices in $T$. For full credit, your algorithm should run in $O(n^2 \log n)$ time. [Hint: Use your algorithm for part (a) as a subroutine. You may assume your algorithm for part (a) is correct and runs in $O(nr)$ time.]

3. Describe and analyze an algorithm to compute an optimal ternary prefix-free code for a given array of frequencies $f[1..n]$. In other words, each character in the alphabet should be assigned a string of 0s, 1s, and 2s such that no code is a prefix of any other and the total length of the encoded message represented by the frequencies is as small as possible. Similar to what we saw in lecture, your output can be the representation of a ternary code tree. See Figure 4.4 of Erickson 4 and the preceding text for an example of how to build an optimal binary code tree.

For simplicity, you may consider the empty string to be a valid code word for the case $n = 1$. Don't forget to prove that your algorithm is correct for all $n$.

4. There are $n$ galaxies connected by $m$ intergalactic teleport-ways. Each teleport-way joins two galaxies and can be traversed in both directions. Also, each teleport-way $e$ has an associated cost $c(e)$ of dollars, where $c(e)$ is a positive integer. A teleport-way can be used multiple times, but the toll must be paid every time it is used.

Judy wants to travel from galaxy $s$ to galaxy $t$, but teleportation is not very pleasant and she would like to minimize the number of times she needs to teleport. However, she wants the total cost to be a multiple of five dollars, because carrying small change is not pleasant either.

Describe and analyze an algorithm to compute the smallest number of times Judy needs to teleport to travel from galaxy $s$ to galaxy $t$ so that the total cost is a multiple of five dollars. To emphasize, Judy wants to minimize the number of times she teleports. Any total cost is fine as long as it is a multiple of five dollars.

[Hint: Build a graph that can model Judy’s location and how much small change she has. Then, run an appropriate graph search algorithm that you’ve seen before.]