\( T(n) = \max_{1 \leq r \leq n} \left( T(r-1) + T(n-r) \right) + \Theta(n) \)

\[ T(n) \geq c \cdot n + (n-1) + (n-2) + \ldots \]
\[ = \Omega(n^2) \]
worst case really is in $\Theta(n^2)$

Usually, things are better. Subproblems are same size and recurrence is more like

$$T(n) \leq T(n/3) + T(2n/3) + \Theta(n)$$

if $r \in \left[\frac{n}{3}, \frac{2n}{3}\right]$
\# levels \leq \log_{3/2} n = O(\log n)
T(n) = O(n \log n)

\# full levels \geq \log_2 n = \Omega(\log n)
T(n) = \Omega(n \log n)
\implies T(n) = \Theta(n \log n)
Median Selection

(in this class) median: the element at index \( \lceil n/2 \rceil \) after sorting

Hoare '61: (same paper as quicksort)

pivot

\[
\text{rank } r = \lceil n/2 \rceil \text{ so search left side.}
\]

need recursive calls to accept a rank as a parameter
Selection (formally): Given $A[1..n] + k$, return element of rank $k$ (lies in position $k$ in sorted order)

**QuickSelect**($A[1..n], k$):

if $n = 1$

   return $A[1]$

else

   Choose a pivot element $A[p]$

   $r \leftarrow$ PARTITION($A[1..n], p$)

   if $k < r$

      return QuickSelect($A[1..r-1], k$)

   else if $k > r$

      return QuickSelect($A[r+1..n], k-r$)

   else

      return $A[r]$
**Worst-case**

\[ T(n) = \max_{1 \leq l \leq n-1} T(l) + \Theta(n) \]

\[ T(n) = \Theta(n^2) \quad \text{if} \quad l = n-1 \]

Each time

Suppose \( l = an \) for some constant \( a < 1 \)

\[ T(n) \leq T(an) + \Theta(n) \]
decreasing geometric...

\[ T(n) = \Theta(n) \]
Median of Medians

BFPRT '70s.

Divide input into $\lceil n/5 \rceil$ blocks of 5 elements each. (Assume $n \leq 5$)

Find median of each block.

Find median of these medians (mom) using our algorithm recursively.

Use mom as the pivot.
\textbf{MomSelect}(A[1..n], k):

moves rank \( k \) element to position \( k \) \& returns it.

\begin{verbatim}
MomSelect(A[1..n], k):
    if \( n \leq 25 \) \texttt{(or whatever)}
        use brute force
    else
        m \leftarrow \lceil n/5 \rceil
        for \( i \leftarrow 1 \) to \( m \)
            \textbf{MedianOfFive}([A[5i-4..5i]]) \texttt{(Moves median to index 5i - 2.)}
            swap A[i] \leftrightarrow A[5i - 2]
            MomSelect(A[1..m],[m/2]) \texttt{(Recursion! Also, moves mom to index \lfloor m/2 \rfloor.)}
        r \leftarrow \textbf{Partition}(A[1..n],[m/2])
        if \( k < r \)
            return MomSelect(A[1..r-1], k) \texttt{(Recursion!)}
        else if \( k > r \)
            return MomSelect(A[r+1..n], k-r) \texttt{(Recursion!)}
        else
            return A[r]
\end{verbatim}
Analysis

Imagine...

Lay out $A$ as a $5 \times \lceil n/5 \rceil$ grid.
Each column is one of the blocks of 5.
Sort each column
Sort collection of columns by their medians.
\[ \text{symmetry, } \frac{10}{\sqrt{n}} \leq \frac{10}{\sqrt{n}} \rightarrow \frac{10}{\sqrt{n}} < \frac{3}{\sqrt{n}} \leq \frac{10}{\sqrt{n}} \rightarrow \frac{10}{\sqrt{n}} < \frac{3}{\sqrt{n}} \leq \frac{10}{\sqrt{n}} \rightarrow \text{ median within a block} \rightarrow \text{median of each block} \rightarrow \text{median} \rightarrow \left[ \left\lfloor \frac{n/5}{2} \right\rfloor \times \text{block} \right] \]
So second call on $e \geq \frac{2n}{10}$ elements.

$$T(n) = T\left(\frac{n}{5}\right) + T\left(\frac{7n}{10}\right) + \Theta(n)$$

- $e = n$
- $e = \frac{9n}{10}$
- $e = \frac{81n}{100}$

Level: sum up $\left(\frac{9}{10}\right)^i n$

$$T(n) = \Theta(n)$$
Why is 5 odd?

So, the second call would have size \( \frac{2n}{3} \), and

\[
T(n) = T(\frac{n}{3}) + T(\frac{2n}{3}) + f(n\log n)
\]

So, 3?

5 is odd.

Why 5?
Don't use Mom Select!

Pick a pivot uniformly at random.

For both Quick Sort and Select,

Expected time is $O(n \log n)$