Given directed $G = (V, E, w)$ with $w : E \rightarrow \mathbb{R}$, the shortest path from $s \in V$ to $t \in V$ is the path $P$ that minimizes $w(P) = \sum_{u \rightarrow v \in P} w(u \rightarrow v)$. The minimum of $w(P)$ is the distance from $s$ to $t$.

Most shortest paths algorithms find shortest paths from a single source $s$ to all vertices $v$. **Single-Source Shortest Paths**
Shortest paths contain shortest paths

Can assume shortest paths follow a tree rooted at s.

Single-source shortest paths tree
MST ≠ DSSP

For undirected graphs, replace each edge with both of its orientations.

\[ \text{to make graph directed.} \]
Weights may be negative!

We don't know how to find actually shortest paths.

We can find shortest walks.

Shortest walks don't exist if there is a negative weight cycle.
o.w. some shortest walk is a path & SSSP tree exists
Ford, Dantzig, Minty...

Maintain a pessimistic guess on distance to each vertex.

Two mutable variable per vertex v:

- \( \text{dist} (v) \): upper bound on distance from s to v.
  - \( \text{dist}(s) \) starts at 0
  - \( \text{dist}(v) \) starts at +\( \infty \), \( \forall v \neq s \)

- \( \text{pred}(v) \): predecessor of v on some s to v walk.
  - \( \text{pred}(v) \) starts at Null
InitSSSP(s):
  \[ \text{dist}(s) \leftarrow 0 \]
  \[ \text{pred}(s) \leftarrow \text{Null} \]
  for all vertices \( v \neq s \)
  \[ \text{dist}(v) \leftarrow \infty \]
  \[ \text{pred}(v) \leftarrow \text{Null} \]

Call an edge \( u \rightarrow v \) tense if
\[ \text{dist}(u) + w(u \rightarrow v) \leq \text{dist}(v) \]
implies \( \text{dist}(v) \) is too high,
so relax it.

\[
\text{RELAX}(u \rightarrow v):
\]
\[ \text{dist}(v) \leftarrow \text{dist}(u) + w(u \rightarrow v) \]
\[ \text{pred}(v) \leftarrow u \]

The "one" algorithm:

\[
\text{FordSSSP}(s):
\text{InitSSSP}(s)
\]
while there is at least one tense edge
\[ \text{RELAX any tense edge} \]
Claims: Eventually terminates if no negative cycles.

When it does $\text{dist}(v) = \text{distance to } v \neq v$, $\text{pred}(v) = \text{pred vertex on shortest path}$

if $\text{dist}(v) = \infty$, $v$ is not reachable from $s$.

But...

if any neg. weight cycle is reachable from $s$, we'll never terminate.
Lemma: At all times, for any vertex \( v \), \( \text{dist}(v) \) is either \( \infty \) or the length of some \( s,v \)-walk that ends with \( \text{pred}(v) \rightarrow v \).

Proof: Induction on \( A \) relaxations.

Suppose \( \text{dist}(v) \neq \infty \) and the last change was \( \text{Relax}(u \rightarrow v) \).

We have \( \text{dist}(v) \leq \text{dist}(u) + w(u \rightarrow v) \) and \( \text{pred}(v) \leq u \).

\( \text{dist}(u) \) was last set at an earlier relaxation.
So there is a walk $W$ from $s$ to $u$ such, $w(W) = v_{\text{dist}(u)}$.

We just set $\text{dist}(v)$ to be the length of $W$, $u \rightarrow v$, which ends with $u \rightarrow v = \text{pred}(v) \rightarrow v$.

Cori $\Rightarrow$ $\text{dist}(v)$ is always $\geq$ distance from $s$ to $v$. 
Never a Bad Idea: Bellman-Ford

BellmanFord(s)
InitSSSP(s)
while there is at least one tense edge
for every edge $u \rightarrow v$
if $u \rightarrow v$ is tense
RELAX($u \rightarrow v$)

Relax ALL the edges! (over and over)

Imagine Ford sssp. $\text{dist}(v)$ is correct first time we relax $s \rightarrow v$.

First time we relax $v \rightarrow w$ after setting $\text{dist}(w)$, we're done.
First time we relax \( w \rightarrow x \), after we relax \( v \rightarrow w \) after we relax \( s \rightarrow v \), \( \text{dist}(x) \) is correct.

\[
\text{dist}^i(v): \text{length of shortest walk from } s \text{ to } v \text{ that uses at most } i \text{ edges.}
\]

- \( \text{dist}^0(s) = 0 \)
- \( \text{dist}^0(v) = \infty \) \( \forall v \neq s \)
- if the shortest walk from \( s \) to \( v \) uses \( k \) edges, \( \text{dist}^k(v) > \text{distance } \) if \( i < k \)
  \[
  \text{dist}^i(v) = \text{distance } \text{ if } j \geq k
  \]
Lemma: For every vertex \( v \) and non-negative \( \omega \), after \( \omega \) iterations of outer while loop, \( \text{dist}(v) \leq \text{dist}_{\in \omega}(v) \). 

Proof: If \( \omega = 0 \), \( \text{dist}(s) = \text{dist}_{\in 0}(s) \); \( \text{dist}(v) = \infty = \text{dist}_{\in 0}(v) \).

Suppose \( \omega > 0 \).

Let \( W \) be a shortest walk from \( s \) to \( v \) using \( \in \omega \) edges. By def. \( W \) has length \( \text{dist}_{\in \omega}(v) \).

If \( W = \emptyset \), then \( v = s \) then \( \text{dist}_{\in \omega}(v) = 0 \).
dist(v) = dist(s) = 0 = dist_i(v)

\text{o.w. Let } u \rightarrow v \text{ be the last edge of } W. \\

\text{By induction, } dist(u) = dist_{\hat{u}-1}(u) \text{ after } \hat{u}-1 \text{ iterations.}

\text{During iteration } \hat{u}, \text{ we check } u \rightarrow v. \text{ At that moment}
a) \( \text{dist}(v) \leq \text{dist}(u) + w(u \rightarrow v) \)

or-

b) \( \text{dist}(v) > \text{dist}(u) + w(u \rightarrow v) \)

\[ \Rightarrow u \rightarrow v \text{ is tense} \]

\[ \Rightarrow \text{we relax it so now} \]

\[ \text{dist}(v) = \text{dist}(u) + w(u \rightarrow v) \]

\[ \text{dist}(v) \text{ doesn't rise}, \text{ so when} \]

iteration \( i \) ends,

\[ \text{dist}(v) \leq \text{dist}(u) + w(u \rightarrow v) \]

\[ \leq \text{dist}^i (u) + w(u \rightarrow v) \]

\[ \leq \text{dist}^i (u) - i \]

\[ = \text{dist}^i_i (v) \]

\[ \text{Proof works even with neg. cycles!} \]
Suppose no neg. cycles...

Shortest walks/paths have \( \leq |V| - 1 \) edges.

\[
\Rightarrow \text{dist}(v) = \text{distance after } |V| - 1 \text{ iterations}
\]

(we know \( \text{dist}(v) \geq \text{distance too...} \))

\[
\Rightarrow \text{dist}(v) = \text{distance}
\]

\( O(E) \) per iteration, so \( O(VE) \) time
Fail safe version:

$$\text{BellmanFord}(s)$$
$$\text{InitSSSP}(s)$$
repeat $V-1$ times
  for every edge $u \rightarrow v$
    if $u \rightarrow v$ is tense
      $$\text{RELAX}(u \rightarrow v)$$
  for every edge $u \rightarrow v$
    if $u \rightarrow v$ is tense
      return “Negative cycle!”

still $O(VE)$ time.