SECRET RM-1573 10-24-55 -33-

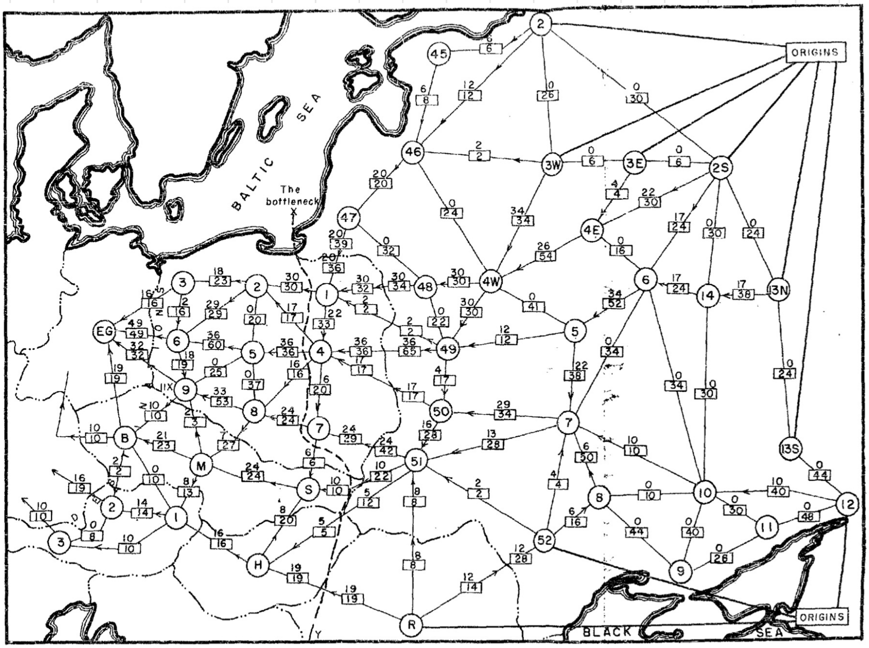


Fig. 7 — Traffic pattern: entire network available

Legend:

—··— International boundary

(8) Railway operating division

$\overset{9}{\boxed{12}}$ Capacity: 12 each way per day. Required flow of 9 per day toward destinations (in direction of arrow) with equivalent number of returning trains in opposite direction

All capacities in $\left.\begin{matrix}\text{trains}\\ \sqrt{1000\text{'s of tons}}\end{matrix}\right\}$ each way per day

Origins: Divisions 2, 3W, 3E, 2S, 13N, 13S, 12, 52 (USSR), and Roumania

Destinations: Divisions 3, 6, 9 (Poland); B (Czechoslovavakia); and 2, 3 (Austria)

Alternative destinations: Germany or East Germany

Note IIX at Division 9, Poland

Two problems:

Given directed graph $G = (V, E)$.

s : source ← two vertices

t : target/sink

Maximum Flow

An (s,t)-flow is an assignment $f: E \to \mathbb{R}_{\geq 0}$ that models how material flows through the network.

It must follow the <u>conservation constraint</u>:

for each vertex v (except maybe s & t)

$$\sum_{w} f(v \to w) = \sum_{u} f(u \to v)$$

(we'll say $f(v \to w) = 0$ if $v \to w \notin E$)

$$\delta f(v) := \sum_w f(v \to w) - \sum_u f(u \to v)$$

so $\delta f(v) = 0$ if $v \notin \{s, t\}$

The <u>value</u> of f is

$$|f| := \delta f(s) = \sum_w f(s \to w) - \sum_u f(u \to s)$$

Claim: $\delta f(s) = -\delta f(t)$

Proof:

$$0 = \sum_v \delta f(v) = \delta f(s) + \delta f(t)$$

Edges have <u>capacities</u>

$$c : E \to \mathbb{R}_{\geq 0}$$

Flow f is **feasible** with respect to capacities c if $f(e) \le c(e)\ \forall e \in E$

Say a flow f...

saturates edge e if $f(e) = c(e)$

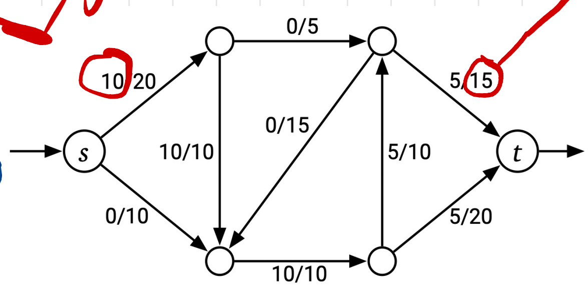
avoids e if $f(e) = 0$

$G + s + t + c$: **flow network**

flow on edge

capacity

$|f| = 10$



maximum flow problem:

find a feasible (s,t)-flow of max value

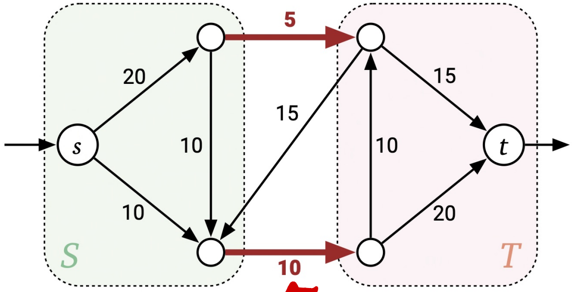
Minimum Cut

An <u>(s,t)-cut</u> is a partition of V into disjoint S & T.
(so $S \cup T = V$ & $S \cap T = \emptyset$)
where $s \in S$ & $t \in T$.

The <u>capacity</u> of cut (S,T) is the sum of capacities for edges going <u>from</u> S <u>to</u> T.

$$\|S,T\| := \sum_{v \in S} \sum_{w \in T} c(v \to w)$$

(we'll say $c(u \to v) = 0$ if $u \to v \notin E$)



capacity

$\|S, T\| = 15$

minimum cut problem:
find an (s,t)-cut of min capacity

Lemma: The value of any feasible (s,t)-flow f is at most the capacity of any (s,t)-cut (S,T).

$$
\begin{aligned}
|f| &= \partial f(s) \\
&= \sum_{v \in S} \partial f(v) \\
&= \sum_{v \in S} \sum_{w} f(v \to w) - \sum_{v \in S} \sum_{u} f(u \to v) \\
&= \sum_{v \in S} \sum_{w \notin S} f(v \to w) - \sum_{v \in S} \sum_{u \notin S} f(u \to v) \\
&= \sum_{v \in S} \sum_{w \in T} f(v \to w) - \sum_{v \in S} \sum_{u \in T} f(u \to v)
\end{aligned}
$$

$$\leq \sum_{v \in S} \sum_{w \in T} f(v \to w) \qquad [f \geq 0]$$

$$\leq \sum_{v \in S} \sum_{w \in T} c(v \to w)$$

$$= \|S, T\|$$

$|f| = \|S, T\|$ iff we avoid all edges from T to S & saturate all edges from S to T

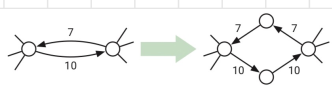
Max-Flow Min-Cut Theorem
[Ford + Fulkerson '54]
([Elias, Feinstein, Shannon '56]):

The maximum flow value = the min cut capacity.

We'll assume G is reduced.

For every pair $u, v \in V$, we have at most one of $u \to v$ or $v \to u$.

Can guarantee:



Proof: Let f be an arbitrary feasible (s,t)-flow.

Either we can find a better flow or $|f| = \|S,T\|$ for some (S,T).

The residual capacity function $c_f : V \times V \to \mathbb{R}$

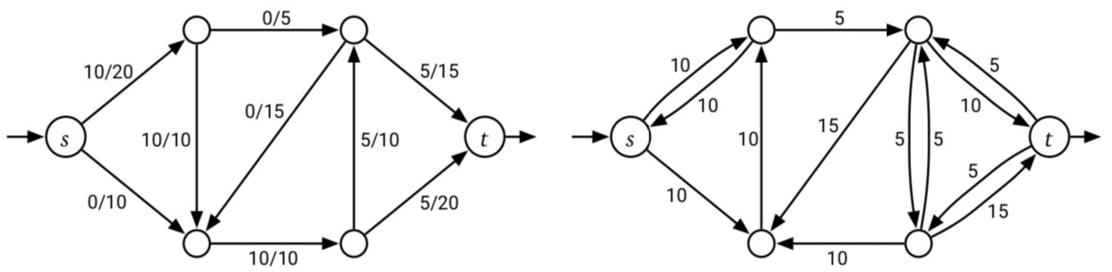
$$c_f(u \to v) = \begin{cases} c(u \to v) - f(u \to v) & \text{if } u \to v \in E \\ f(v \to u) & \text{if } v \to u \in E \\ 0 & \text{o.w.} \end{cases}$$

$f(u \to v) \geq 0$ & $f(u \to v) \leq c(u \to v)$

so res. caps are non-neg.

Residual graph $G_f = (V, E_f)$

E_f: all $u \to v$ s.t. $c_f(u \to v) > 0$



G $\qquad$ G_f

Either there is a path P from s to t (in G_f) or there isn't...

Suppose P exists...

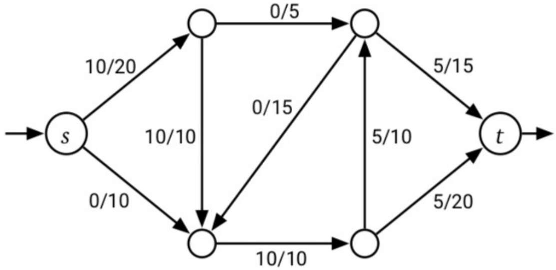
Call it an <u>aumenting path</u>.

Let $F := \min_{u \to v \in P} c_f(u \to v)$

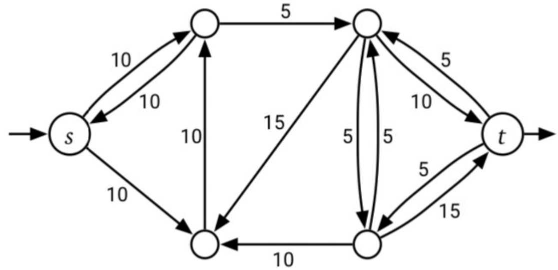
We'll "push" F units of flow along P.

Define a new flow $f' : E \to \mathbb{R}$ where $f'(u \to v) =$

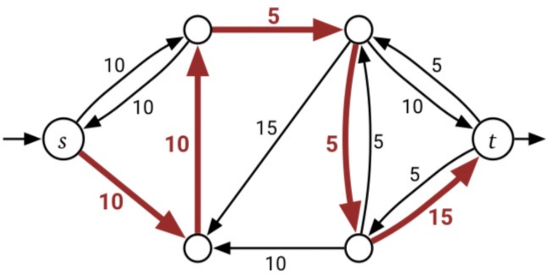
$$\begin{cases} f(u \to v) + F & \text{if } u \to v \in P \\ f(u \to v) - F & \text{if } v \to u \in P \\ f(u \to v) & \text{o.w.} \end{cases}$$



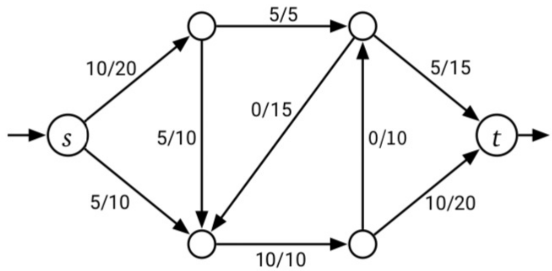
f



G_f



P



f'

Facts: f' is feasible.

$$|f'| = |f| + F$$

So f was not max value.

Suppose P does not exist.

S : vertices reachible from s in G_f

$T := V \setminus S$.

(S, T) is an (s,t)-cut.

For all $u \in S$ & $v \in T$,
$u \to v \notin E_f$.

If $u \to v \in E$, $0 = c_f(u \to v)$
$= c(u \to v) - f(u \to v)$

so $u \to v$ is <u>saturated</u>

If $v \to u \in E$, $0 = c_f(u \to v)$
$= f(v \to u)$

so $v \to u$ is <u>avoided</u>

f saturates all $S \to T$ edges & avoids all $T \to S$ edges

So $|f| = \|S, T\|$.

So f is max value &
(S, T) is min capacity.