Given flow network:

Directed graph $G = (V, E)$

(reduced: for any $u, v \in V$ there is at most one of $u \to v$ or $v \to u$ in E

$s, t \in V$

capacities $c: E \to \mathbb{R}_{\geq 0}$

Max-flow Min Cut Proof:

Take any feasible (s,t)-flow

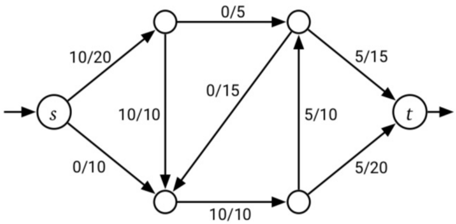
$f : E \to \mathbb{R}_{\geq 0}$.

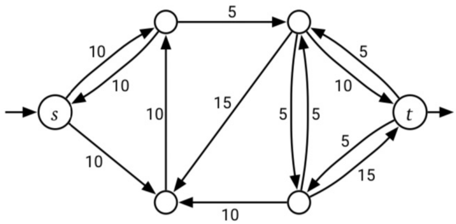
residual capacities $c_f : V \times V \to \mathbb{R}_{\geq 0}$

$$c_f(u \to v) = \begin{cases} c(u \to v) - f(u \to v) & \text{if } u \to v \in E \\ f(v \to u) & \text{if } v \to u \in E \\ 0 & \text{o.w.} \end{cases}$$

$G_f = (V, E_f)$. E_f all $u \to v$ s.t.

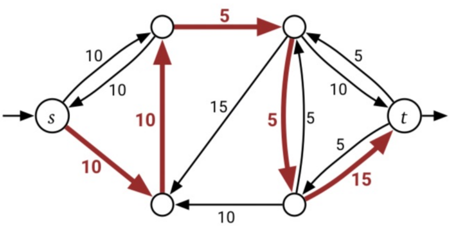
$c_f(u \to v) > 0$

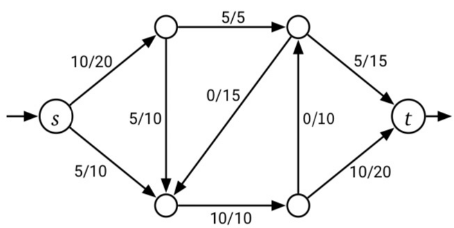




G & f/c G_f & c_f

a) if there is an (s,t)-path P:





G_f & c_f & P G & f'/c

$|f'| > |f|$ by min res. cap. on P

b) o/w.

S: vertices reachible from s in G_f

$T := V \setminus S$

$$\Rightarrow |f| = \|S, T\|$$

max flow

min cut

Ford-Fulkerson Augmenting Path Algorithm:

Start with $f \leftarrow$ all 0 flow

$(f(e) = 0 \;\; \forall e \in E)$

$(|f| = 0)$

Repeatedly look for an augmenting path in G_f, update f to new flow, & recurse.

Analysis: Assume c is integral
(uses integer values)

f starts all 0's (integral)

f remains integral after
each push

f increases by $\geq \dots 1$

Let f^* be a maximum
flow.

So $\leq |f^*|$ iterations

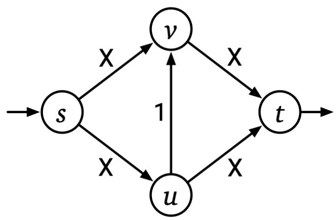
$O(E)$ per iteration

Running time: $O(E|f^*|)$

in worst case.

pseudopolynomial time :

polynomial in numbers involved



Could take $2X$ iterations.

But graph has a representation of size $\Theta(\log X)$, so $O(X)$ is exponential in input size.

FF alg probably faster in practice:

1) $|f^*|$ may be small
2) We have to be pretty unlucky to hit that time bound

With real capacities, alg may not terminate or even approach max flow value in the limit!

Flow properties.

Let f & g be two (s,t)-flows.

Let α & β be real numbers.

Let $h: E \to \mathbb{R}$

$$h(u \to v) := \alpha \cdot f(u \to v) + \beta \cdot g(u \to v)$$

Claim: h is a (s,t)-flow

(possibly negative on some edges.

may not be feasible)

$$|h| = \alpha \cdot |f| + \beta \cdot |g|$$

Let P be any (s,t)-path
(maybe using some edges
backwards)

<u>path flow</u> $P : E \to \mathbb{R}$

$$P(u \to v) = \begin{cases} 1 & \text{if } u \to v \in P \\ -1 & \text{if } v \to u \in P \\ 0 & \text{o.w.} \end{cases}$$

$|P| = 1$

(in augmenting path proof,
$f' \leftarrow f + F \cdot P$.

For any directed cycle C,

<u>cycle flow</u> $C : E \to \mathbb{R}$

$$c(u \to v) = \begin{cases} 1 & \text{if } u \to v \in C \\ -1 & \text{if } v \to u \in C \\ 0 & \text{o.w.} \end{cases}$$

$|c| = 0$

Flow Decomposition Thm:

Every non-negative (s,t)-flow f is the positive linear combination of $\leq |E|$ (s,t)-path & cycle flows that only use edges as directed in E.

Proof sketch:

Take subgraph of positive flow edges.

If $|f| > 0$, walk from s until you repeat a vertex or hit t.

P: path from s to t.

α: min flow value on P

$f \leftarrow f - \alpha \cdot P$

we have one less edge with flow

Can only repeat $|E|$ times.

Consequences:

f is circulation if $|f|=0$.

circulations can be decomposed into $|E|$ weighted cycles

Can make f acyclic but with same value by repeatedly removing flow cycles

Edmonds-Karp 1: Bottleneck Path.

- always use path of max min residual capacity.

Find with variant of Prim-Jarnik

$O(E \log V)$ per iteration

f: current flow

f': max flow $\underline{\text{in } G_f}$

$\Rightarrow f^* = f + f'$

e: bottleneck edge in current iteration

We're about to push $c_f(e)$ units

There is a decomposition of f' is $\leq |E|$ path flows.

Avg path flow has value $|f'| / |E| \Rightarrow c_f(e) \geq |f'|/|E|$

$\Rightarrow$ pushing on path reduces <u>residual</u> max flow value by $|f'|/|E|$ so its now

$\leq (1 - 1/|E|)|f'|$

After $|E| \cdot \ln |f^*|$ iterations

amount left to push is $\leq$

$|f^*| \cdot (1 - 1/|E|)^{E \ln |f^*|}$

$< |f^*| \cdot e^{-\ln |f^*|}$

$= 1$

If capacities are integral, we must be done.

Running time: $O(E^2 \log V \log |f^*|)$

<u>weakly</u> polynomial time

Edmonds-Karp 2: Shortest Augmenting Paths

minimize # edges from s to t in G_f

$O(E)$ per iteration using BFS

Let f_i be flow after i iterations.

Let $G_i := G_{f_i}$

(f_0 : 0 everywhere)
$G_0 = G$)

Let $\text{level}_i(v)$ be unweighted distance from s to v in G_i.

Lemma: $\text{level}_i(v) \geq \text{level}_{i-1}(v)$ for all v & i.

Lemma: Any edge $u \to v$ leaves residual graph at most $|V|/2$ times.

So, $\epsilon \; 2|E||V|$

Runs in $O(VE^2)$ time.

Dinitz '70: $O(V^2E)$

$\vdots$

Orlin '12: $O(VE)$ time

You can cite this running time in HW & exams.