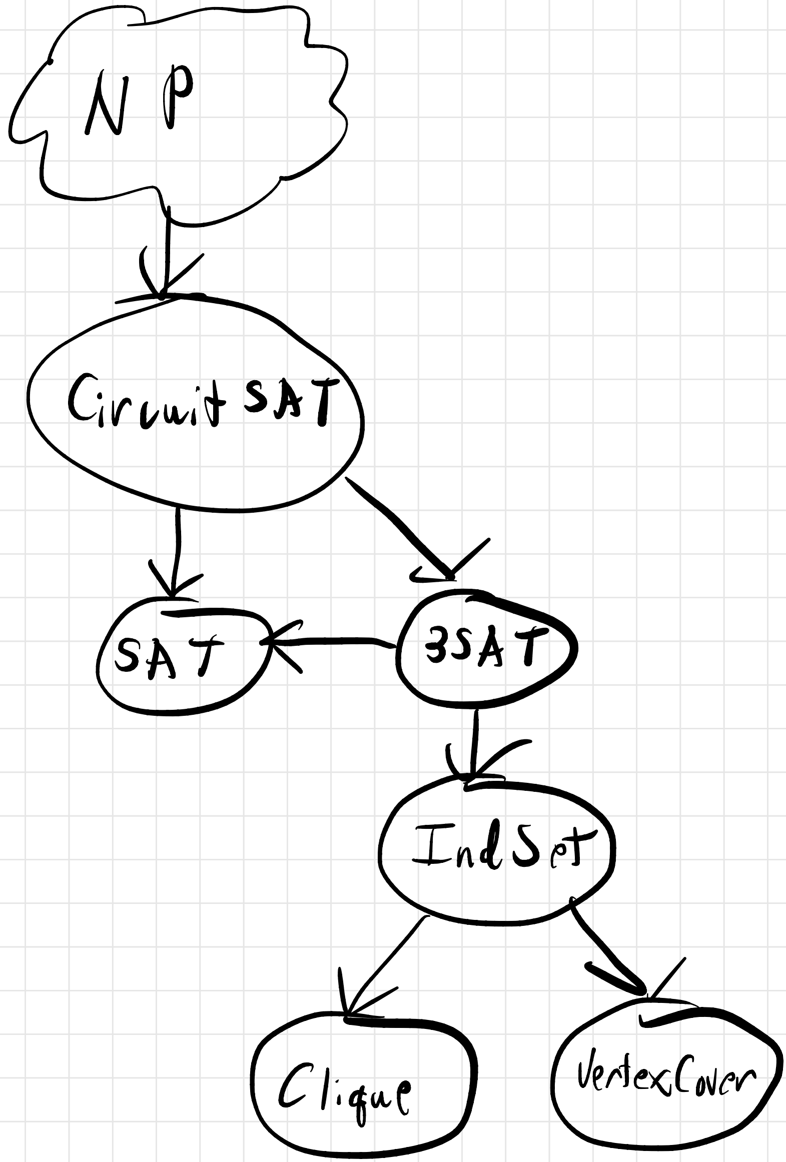

NP
Circuit SAT
SAT
3SAT
IndSet
Clique
VertexCover

<u>proper k-coloring</u> of undirected $G=(V,E)$ is a function $C: V \to \{1, 2, \ldots, k\}$ assigning one of k "colors" to the vertices of G s.t. $C(u) \neq C(v)$ for any edge $uv \in E$.

3Color: Given graph $G=(V,E)$, does it have a proper 3-coloring?

3Color $\in NP$: What is C?

Claim: 3Color is in NP-complete.

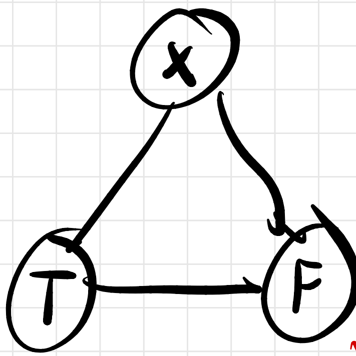
Proof: Reduce from 3SAT.

We're given a 3CNF Φ.

We'll build some <u>gadgets</u>.

1) <u>Truth gadget</u>:

A triangle:

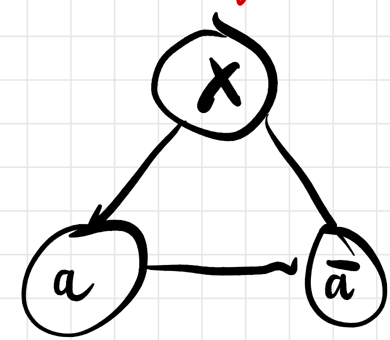


Any proper coloring uses different colors, True, False, & other.

same x as before

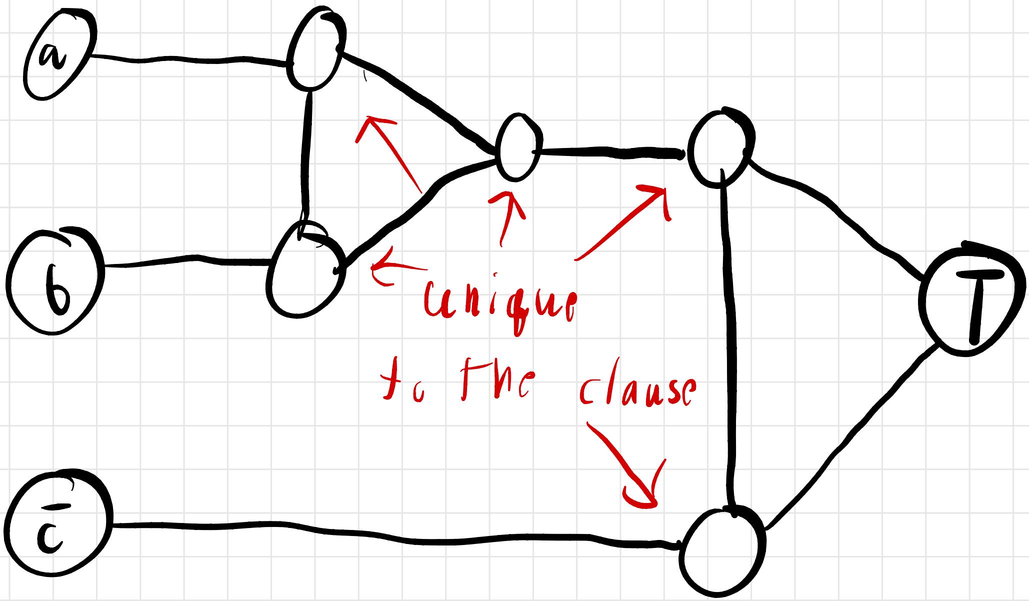
2) Variable gadget:

a $\rightarrow$

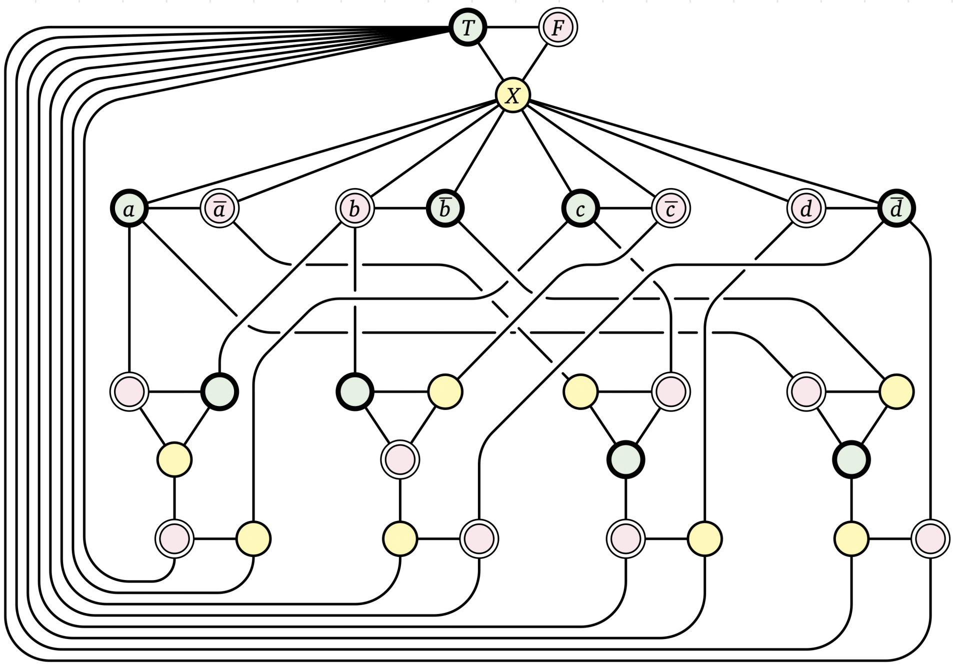


3) Clause gadget: Use the literal vertices from step 2 & vertex T

$(a \vee b \vee \bar{c}) \rightarrow$



Any proper 3-color of the gadet assign's T's color to one or more of a, b, or $\bar{c}$, & vice versa.



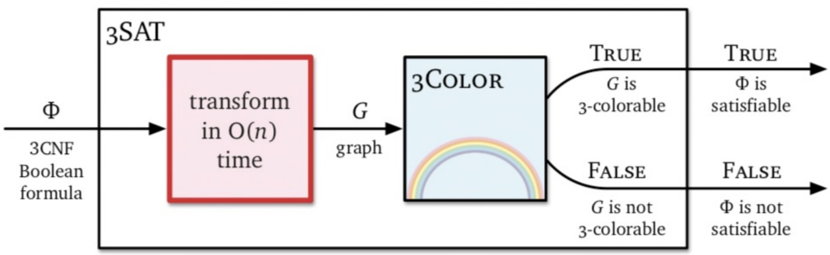
$(a \vee b \vee c) \wedge (b \vee \bar{c} \vee \bar{d}) \wedge (\bar{a} \vee c \vee d) \wedge (a \vee \bar{b} \vee \bar{d})$

That's G.

Claim: Φ is sat. iff G has a proper 3-coloring

Proof: $\Leftarrow$ Use the literals colored as T to sat. Φ

⇒ True literals get T color & False ones get F color. Color clause gadgets as needed.



For any $k \geq 3$, kColor is NP-complete.

MinColor is NP-hard.

A Hamiltonian cycle in a graph visits each vertex exactly once.

(Visit each edge exactly once for an Eulerian Tour)

Directed HamCycle: Given directed $G=(V,E)$. Does G contain a Hamiltonian cycle?

In NP.

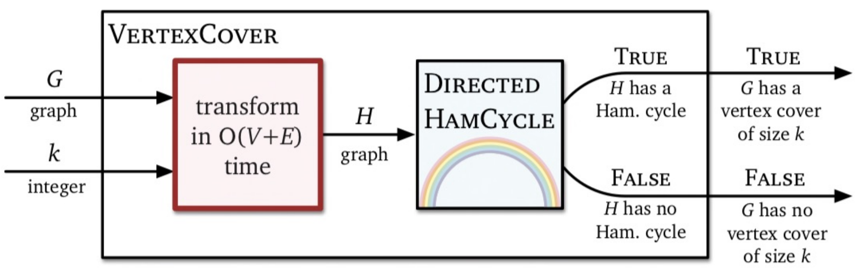
Also NP-complete.

(Eulerian Tour $\in$ P)

Reduce from Vertex Cover.

Given undirected G + an integer k. Is there a vertex cover of size k?

Build graph H...



Edge gadget:

$uv \in G \rightarrow$

four vertices in H

(u, v, in), (u, v, out),
(v, u, in), (v, u, out)

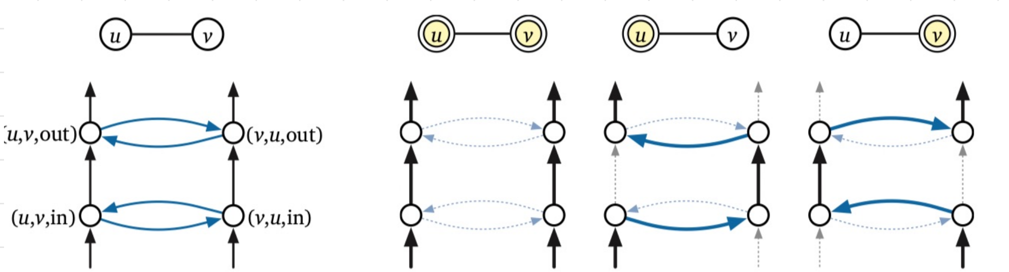
+ six edges in H

$(u, v, in) \rightarrow (u, v, out)$

$(u, v, in) \rightleftarrows (v, u, in)$

$(v, u, in) \rightarrow (v, u, out)$

$(u, v, out) \leftrightarrows (v, u, out)$



Way(s) through describes how to cover uv.

Vertex gadget.

Vertex u in G $\rightarrow$

Say u has d neighbors $v_1, v_2, \dots, v_d$.

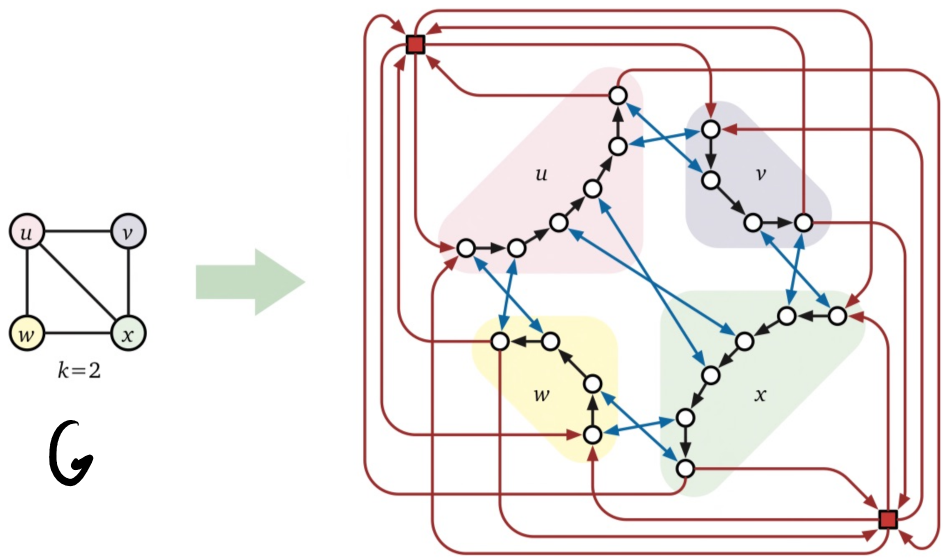
Add edge $(u, v_{i-1}, \text{out}) \rightarrow$ in H (u, v_i, in) for all $2 \le i \le d$.

Called a <u>vertex chain</u>.

Add k <u>cover vertices</u> $x_0, x_1, \dots, x_{k-1}$ + edges

$x_i \to (u, v_1, in)$ for all vertices u,

$(u, v_d, out) \to x_i$



G

H

Suppose there is a vertex cover $u_0, u_1, \ldots, u_{k-1}$.

There is cycle...

For each $i \in \{0, \ldots, k-1\}$

$x_i \to (u_i, v_1, in) \to \ldots$

$\ldots \to (u_i, v_d, out) \to x_{i+1 \ (\mathrm{mod}\ k)}$

Suppose $\exists$ a Ham cycle C...

Undirected HamCycle $\in$ NP-complete

(Un)directed HamPath $\in$ NP-complete

$\Rightarrow$ Longest (Simple) Path is NP-hard

Subset Sum: Given a set X of positive integers & a an integer T. Is there a subset of X summing to T?

in NP

in NP-complete...

Reduce from Vertex Cover.

Given undirected $G=(V,E)$ & integer k.

Edge gadgets: Number edges from 0 to $|E|-1$.

X gets $b_i := 4^i$.

Vertex gadgets:

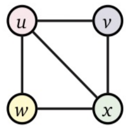
For each vertex v

X gets $a_v := 4^{|E|} + \sum_{i \in \Delta(v)} 4^i$

incident edges

$T := k \cdot 4^{|E|} + \sum_{i=0}^{|E|-1} 2 \cdot 4^i$.

$O(E^2)$ time reduction



$k=2$

$$
\begin{aligned}
a_u &:= 111000_4 = 1344 \qquad & b_{uv} &:= 010000_4 = 256\\
a_v &:= 110110_4 = 1300 & b_{uw} &:= 001000_4 = 64\\
a_w &:= 101101_4 = 1105 & b_{vw} &:= 000100_4 = 16\\
a_x &:= 100011_4 = 1029 & b_{vx} &:= 000010_4 = 4\\
 & & b_{wx} &:= 000001_4 = 1
\end{aligned}
$$

$T = 222222_4$

Suppose G has a vertex cover C of size k.

$X_c := a_v$ for each $v \in V$

$+ b_i$ for each edge covered exactly once

Other direction

But there's a

$O(nT)$ time alg?

pseudo-poly time.

Subset Sum is weakly-NP-hard.

Uses exponentially large numbers.

Other examples were strongly NP-hard.

CS 6382: Theory of Computation

CS 6319: Computational Geometry