NP → Circuit SAT → SAT → 3SAT → IndSet → Clique, VertexCover
**proper k-coloring** of undirected

\[ G = (V, E) \] is a function \( C : V \to \{1, 2, \ldots, k\} \) assigning one of \( k \) "colors" to the vertices of \( G \) such that \( C(u) \neq C(v) \) for any edge \( uv \in E \).

**3Color**: Given graph \( G = (V, E) \), does it have a proper 3-coloring?

3Color \( \in \text{NP} \): What is \( C \)?
Claim: 3Color is in NP-complete.

Proof: Reduce from 3SAT.

We're given a 3CNF $\Phi$.

We'll build some gadgets.

1) Truth gadget:
   A triangle:
   Any proper coloring uses different colors, True, False, and other.

2) Variable gadget:
   $a \quad \rightarrow \quad \overline{a}$
3) Clause gadget: Use the literal vertices from step 2 and vertex $T$.

$(a \lor b \lor \overline{c}) \Rightarrow$ unique to the clause

Any proper 3-color of the gadget assigns $T$'s color to one or more of $a$, $b$, or $\overline{c}$, + vice versa.
That's $G$.

Claim: $G$ is sat. iff $G$ has a proper 3-coloring.

Proof: Use the literals colored as $T$ to sat. $\overline{I}$
True literals get T color & False ones get F color. Color clause gadget as needed.

For any \( k \geq 3 \), \( k\text{-Color} \) is NP-complete. Min Color is NP-hard.
A Hamiltonian cycle in a graph visits each vertex exactly once.

(Visit each edge exactly once for an Eulerian Tour)

Directed Hamiltonian cycle: Given directed $G = (V, E)$. Does $G$ contain a Hamiltonian cycle?

In NP.

Also NP-complete. (Eulerian Tour is in P)
Reduce from Vertex Cover.

Given undirected $G$ and an integer $k$. Is there a vertex cover of size $k$?

Build graph $H$. ...
Edge gadget:

\( u \rightarrow v \in G \Rightarrow \)

Four vertices in \( H \)

\((u, v, \text{in})\), \((u, v, \text{out})\), \((v, u, \text{in})\), \((v, u, \text{out})\)

+ six edges in \( H \)

\((u, v, \text{in}) \rightarrow (u, v, \text{out})\)

\((u, v, \text{in}) \rightarrow (v, u, \text{in})\)

\((v, u, \text{in}) \rightarrow (v, u, \text{out})\)

\((u, v, \text{out}) \leftarrow (v, u, \text{out})\)
Ways through describes how to cover \( uv \).

**Vertex gadget.**

Vertex \( u \) in \( G \) \( \Rightarrow \)

Say \( u \) has \( d \) neighbors \( V_1, V_2, \ldots, V_d \).

Add edge \((u, V_{i-1}, \text{out})\) in \( H \) \( \Rightarrow \) \((u, V_i, \text{in})\) for all \( 2 \leq i \leq d \).
Called a vertex chain.

Add $k$ cover vertices $X_0, X_1, \ldots, X_{k-1}$ and edges $X_i \Rightarrow (u, v, \text{in})$ for all vertices $u, (u, v, \text{out}) \Rightarrow x_i$.
Suppose there is a vertex cover \( u_0, u_1, \ldots, u_{k-1} \).

There is cycle ...

For each \( u \in \{0, \ldots, k-1\} \):

\[ x \rightarrow (w_i, v_j, \text{in}) \rightarrow \ldots \]

\[ \ldots \rightarrow (w_i, v_d, \text{out}) \rightarrow x \]

Suppose \( \exists \) a Hamilton cycle \( C \).

Undirected Hamilton Cycle \( \in \text{NP-complete} \)

(Undirected) Hamilton Path \( \in \text{NP-complete} \)
Longest (Simple) Path is \( \text{NP-hard} \)
Subset Sum: Given a set $X$ of positive integers and an integer $T$. Is there a subset of $X$ summing to $T$?

in NP

in NP-complete...

Reduce from Vertex Cover.
Given undirected graph $G = (V, E)$ and integer $k$. 
Edge gadgets: Number edges from 0 to $|E|-1$. X gets $b_i := 4^i$.

Vertex gadgets:
For each vertex $v$
X gets $a_v := 4^{|E|} + \sum_{\omega \in \Delta(v)} 4^i$.

$T := k \cdot 4^{|E|} + \sum_{i=0}^{1|E|-1} 2 \cdot 4^i$.

$O(E^2)$ time reduction.
Suppose $G$ has a vertex cover $C$ of size $k$.

$X_c := a_v$ for each $v \in V$

$+ b_i$ for each edge $u$ covered exactly once

$$T = \overline{222222}$$

$a_u := 111000_4 = 1344$ \quad $b_{uv} := 010000_4 = 256$

$a_v := 110110_4 = 1300$ \quad $b_{uw} := 001000_4 = 64$

$a_w := 101101_4 = 1105$ \quad $b_{yw} := 000100_4 = 16$

$a_x := 100011_4 = 1029$ \quad $b_{yx} := 000010_4 = 4$

$b_{wx} := 000001_4 = 1$
Other direction

But there's a $O(n^t)$ time alg?

pseudo-poly time.

Subset Sum is weakly-

$NP$-hard.

Uses exponentially large

numbers.

Other examples were strongly

$NP$-hard.
CS 6382: Theory of Computation

CS 6319: Computational Geometry