Fourier Transform (FT)

A signal (x(t)) can be represented in either time domain or spectrum domain.

time-amplitude representation of the signal.

frequency-amplitude representation of the signal.

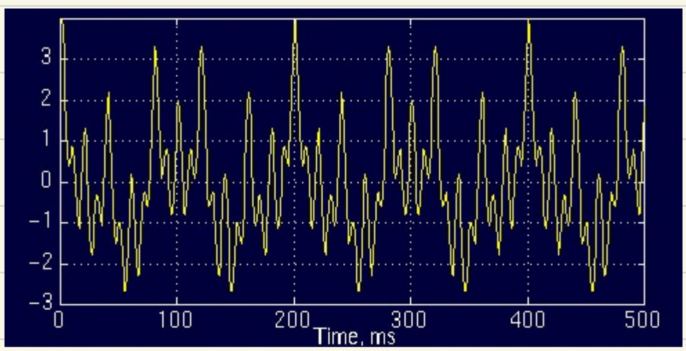
signals
- stationary signals (frequency content do not change in time)
- nonstationary signals (frequency content change in time).

The following is an example of stationary signal:

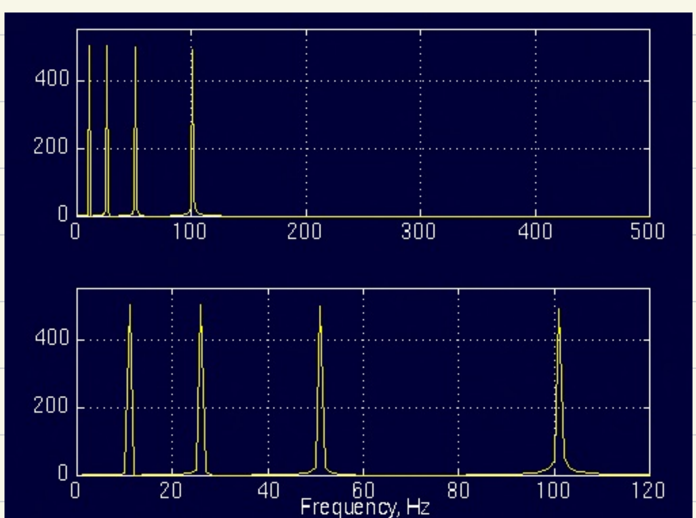
x(t)=cos(2*pi*10*t)+cos(2*pi*25*t)+cos(2*pi*50*t)+cos(2*pi*100*t)

frequencies of 10, 25, 50, and 100 Hz.

time domain

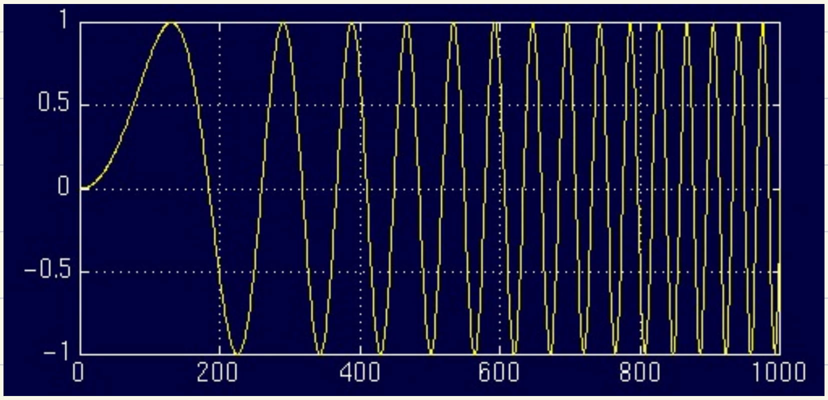


frequency domain

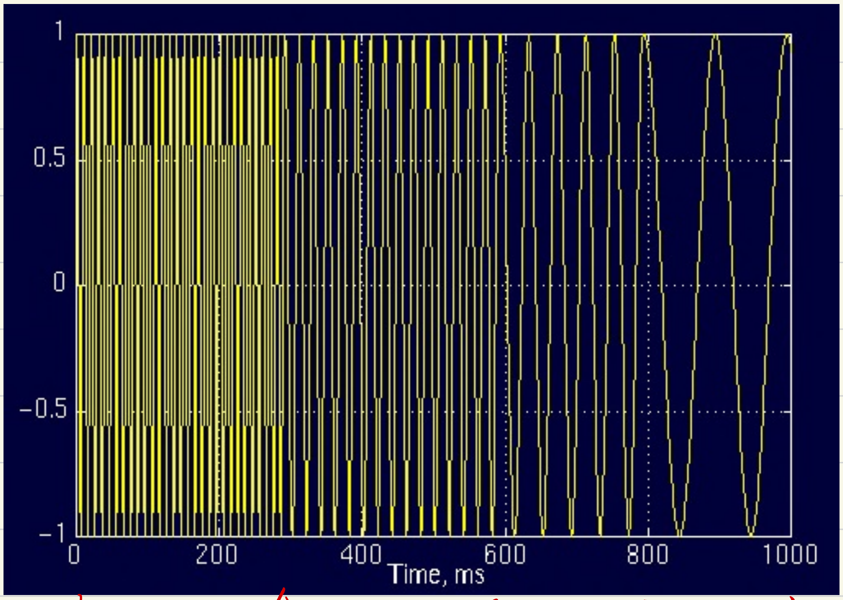


10 25 50 100

The following is an example of nonstationary signal



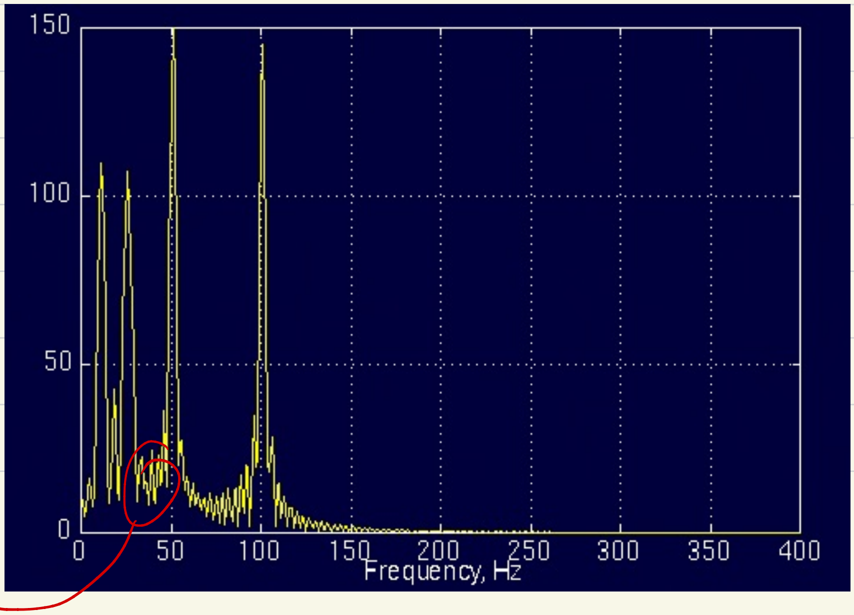
another example:



100 Hz　50 Hz　25 Hz　10 Hz

The following is its FT:

the little ripples are due to sudden changes from one frequency component to another.

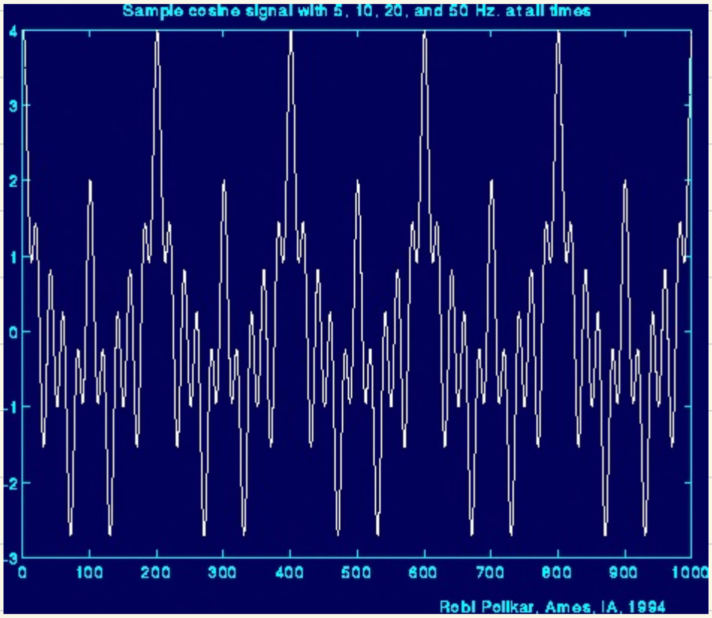


FT gives no information regarding where in time the spectral components appear.

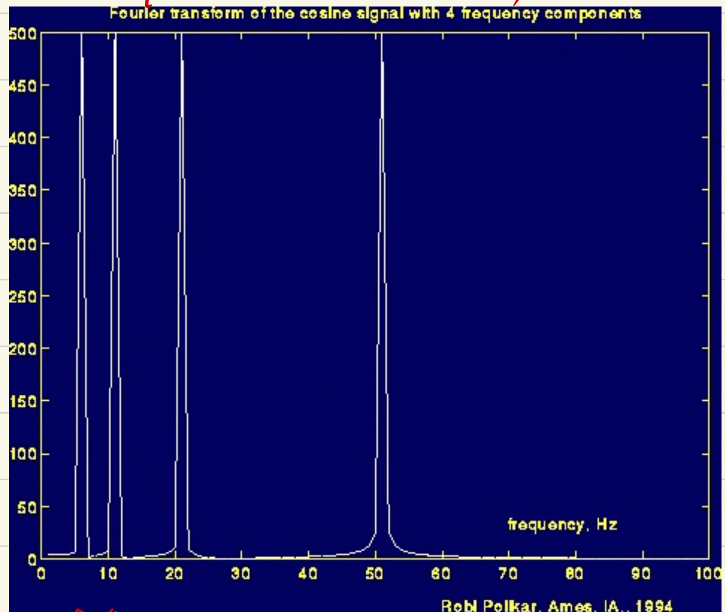
⇓

FT is not a suitable technique for nonstationary signal.

stationary signal (time domain)

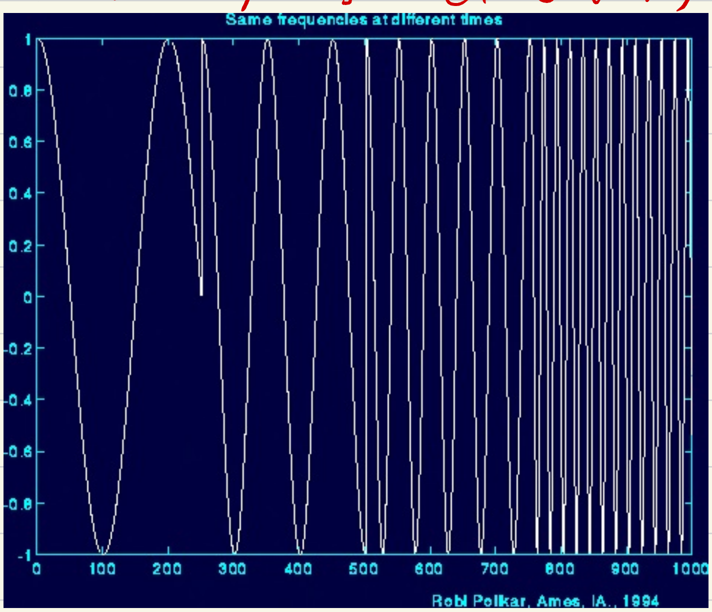


(specturum domain)

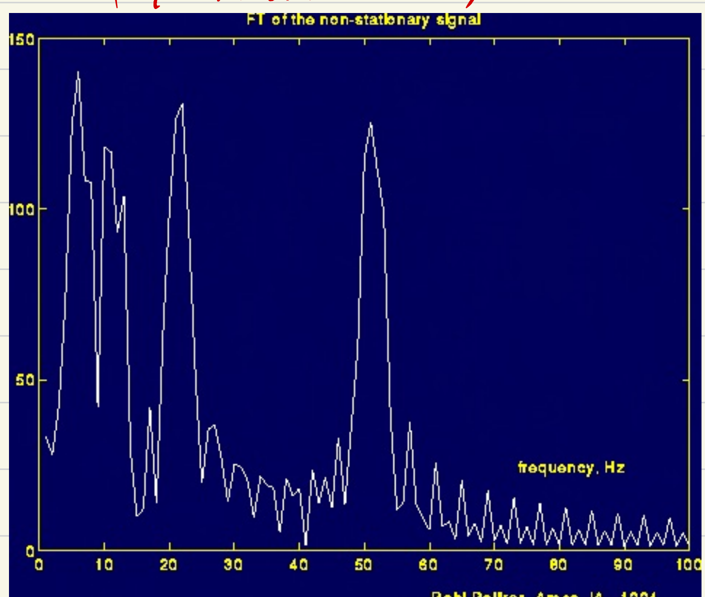


5 10 20 50

nonstationary signal (time domain)

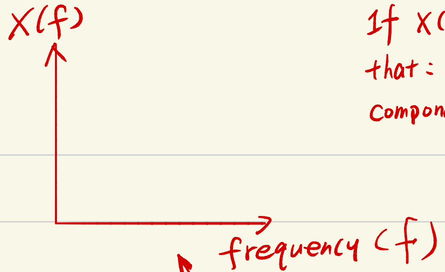


(specturum domain)



5 10 20 50

The above two different signals constitude the same frequency components. That means FT can not distinguish the two signals very well.



If X(f) is a large value, then we say that: the signal x(t), has a dominant spectral component at frequency "f".

Can be written as $\cos(2\pi \cdot ft) + j\sin(2\pi ft)$.

Fourier transform of x(t)

$$\hat{x}(f) = \int_{-\infty}^{\infty} x(t) \cdot e^{-2j\pi f t}\, dt \qquad (1)$$

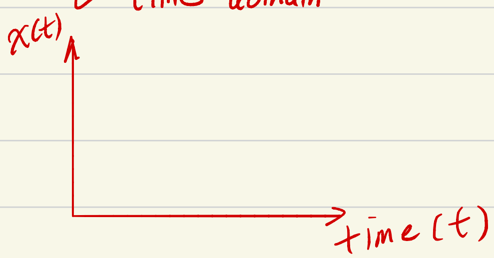
⇒ Fourier transform of x(t).

inverse Fourier transform of X(f).

$$x(t) = \int_{-\infty}^{\infty} \hat{x}(f) \cdot e^{2j\pi f t}\, df \qquad (2)$$

⇒ inverse Fourier transform of X(f)

time domain

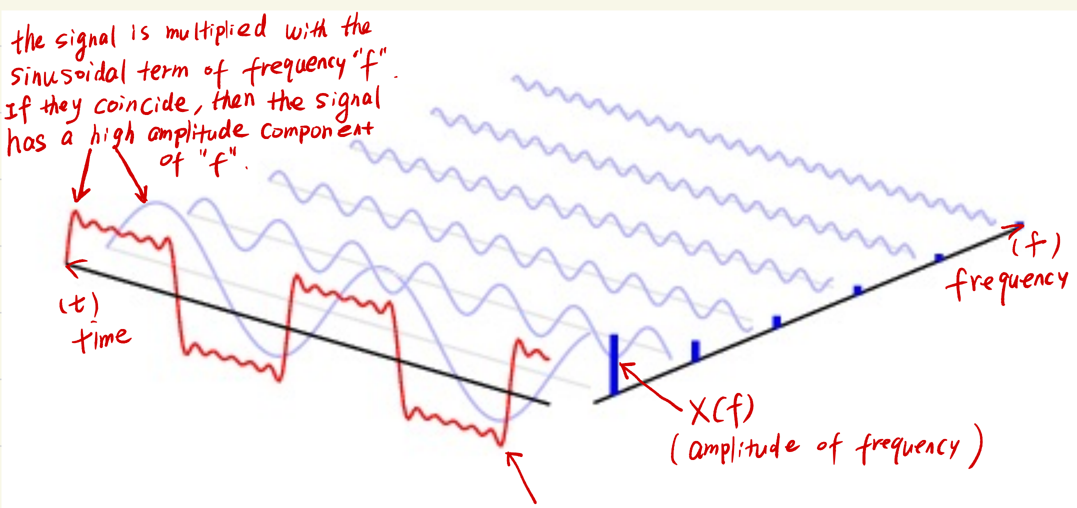


denote $\psi_f(t) = e^{2j\pi f t}$. Then, $\psi_f^*(t) = e^{-2j\pi f t}$

Based on the above notation, we can rewrite the above equations as:

$$\hat{x}(f) = \int_{-\infty}^{+\infty} x(t) \cdot \psi_f^*(t)\, dt$$

$$x(t) = \int_{-\infty}^{\infty} \hat{x}(f) \cdot \psi_f(t)\, df$$



X(t) (amplitude of time)

Convolution theorem

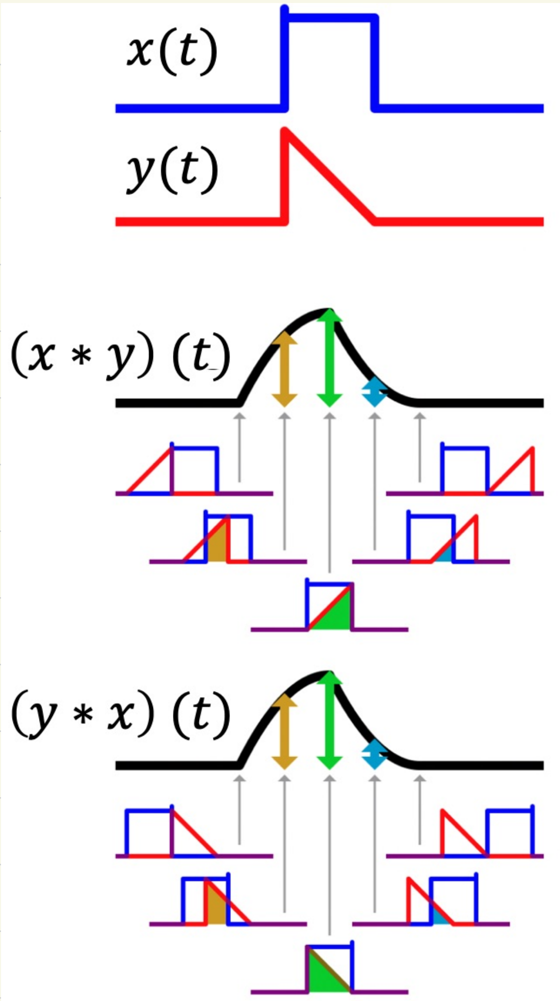
Given two signals $x(t)$ and $y(t)$, the convolution of these two signals is defined as:

$$(x*y)(t) := \int_{-\infty}^{\infty} x(t)\, y(t-\tau)\, d\tau .$$

An equivalent definition is:

$$(x*y)(t) := \int_{-\infty}^{\infty} x(t-\tau)\, y(\tau)\, d\tau$$

That means : $x*y(t) = y*x(t)$.



The convolution theorem based on fourier transform:

$$\boxed{\widehat{(x*y)}(f) = \hat{x}(f)\cdot\hat{y}(f)}$$

This means that the FT of a convolution of two signals is the pointwise product of their FTs.

Scaling property of FT

If you horizontally "Stretch" a signal ($x(t)$) by the factor c in the time domain, you "sequeeze" its FT by the same factor in the frequency domain.

"stretch" the signal by the factor c in the time domain.

$$\boxed{\begin{aligned} x_c(t) &= x(ct) \\ \hat{x}_c(f) &= \hat{x}\left(\frac{f}{c}\right) \cdot / |c| \end{aligned}}$$

"sequeeze" its FT by the same factor in the frequency domain.

The Continuous Wavelet Transform

In FT, the fourier function is denoted as: $\psi_f(t)$, which is dependent only on the frequency (f).
In Wavelet transform (WT), the wavelet function is denoted as: $\psi_{s,\tau}(t)$, which is dependent on both frequency ($f = 1/s$) and time location (τ).

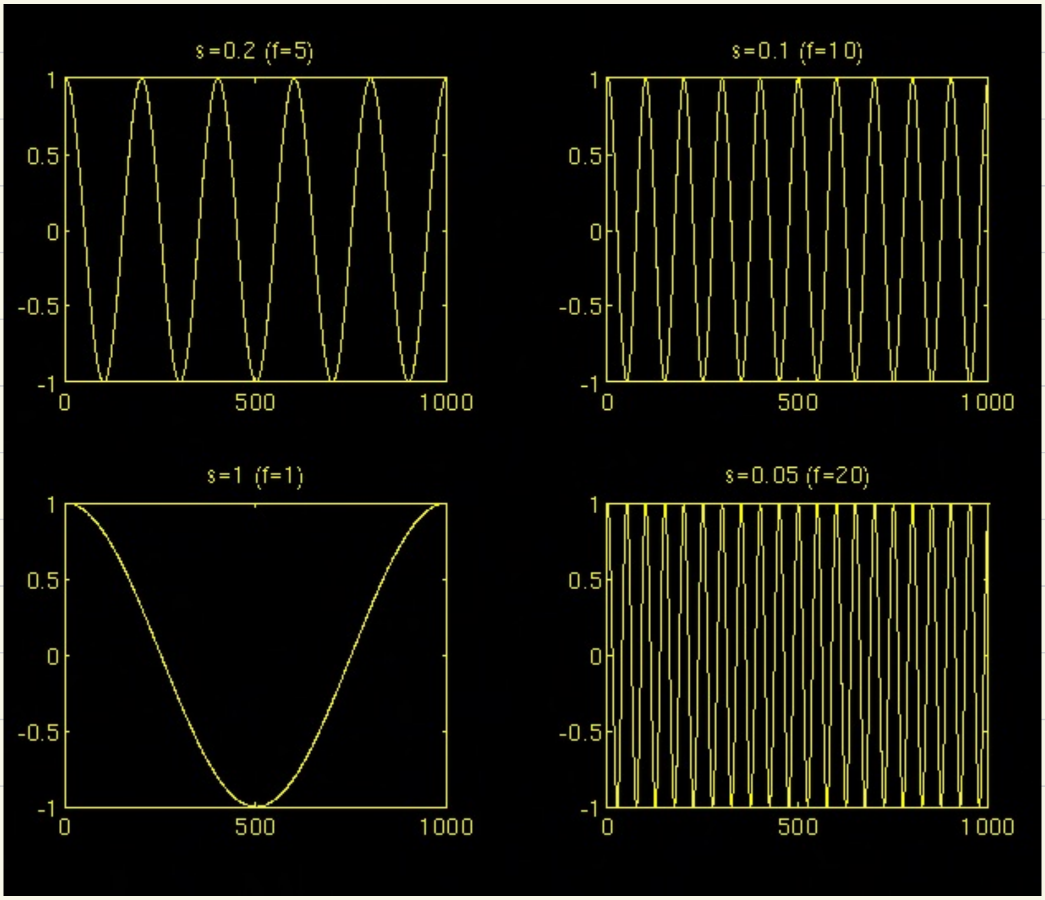
$$CWT_x^{\psi}(\tau, s) = \Psi_x^{\psi}(\tau, s) = \frac{1}{\sqrt{|s|}} \int \underbrace{x(t)}_{\text{signal}} \psi^* \left(\frac{t - \overset{\text{translation}}{\tau}}{\underset{\text{scale}}{s}} \right) dt$$

$$\Big\|$$

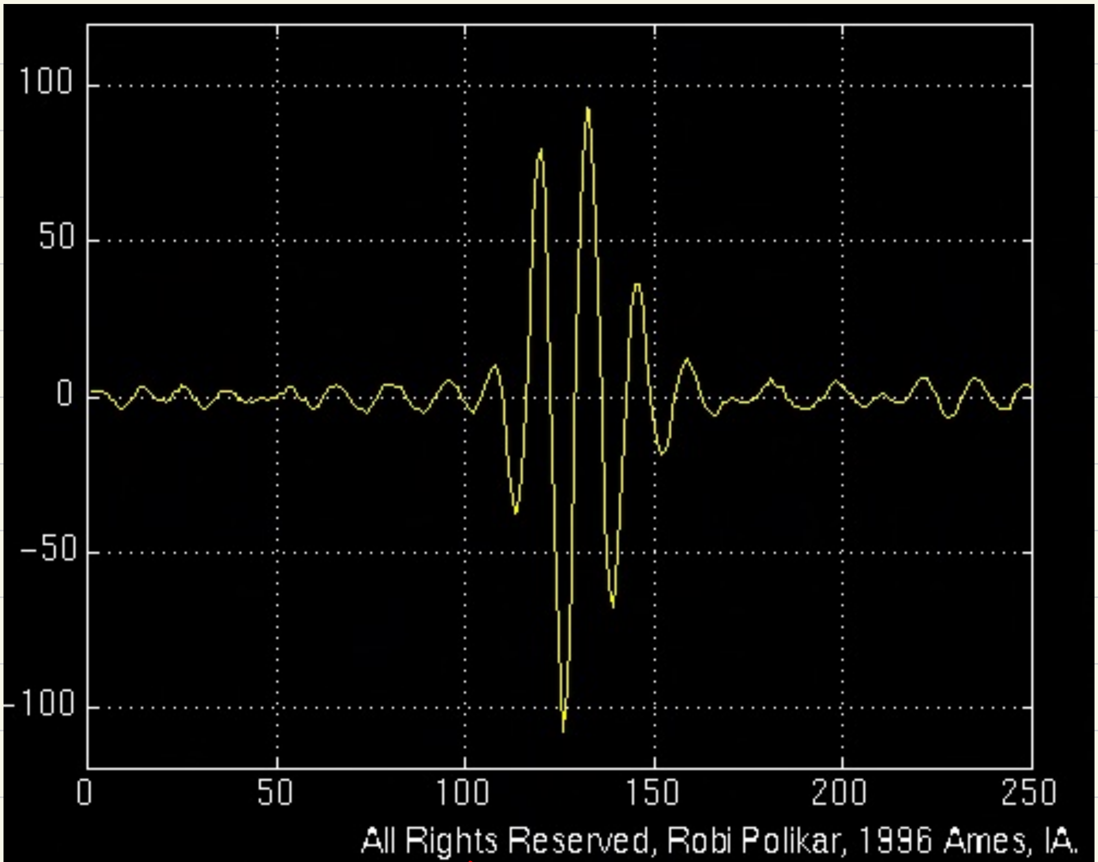
denote $\underbrace{\psi_{s,\tau}(t)}_{\text{Wavelet function}} = \frac{1}{\sqrt{|s|}} \underbrace{\psi}_{\text{mother wavelet function}} \left(\frac{t-\tau}{s} \right)$ — satisfy $\int \psi(t)\, dt = 0$

$$CWT_x^{\psi}(\tau, s) = \Psi_x^{\psi}(\tau, s) = \int x(t) . \psi^*_{\tau,s}(t)\, dt$$

The following are four cosine signals with different scales (frequencies)
stationary signals



The following is a signal that has different scales (frequencies) at different times.



high scale (low frequency) | low scale (high frequency) | high scale (low frequency)

Wavelet transformation

$$\Psi_x^\psi(\tau,s) = \int x(t)\,\psi_{\tau,s}(t)\,dt$$

$$x(t) = \frac{1}{c_\psi^2}\int_s\int_\tau \Psi_x^\psi(\tau,s)\,\frac{1}{s}\,\psi_{\tau,s}(t)\,d\tau\,ds$$

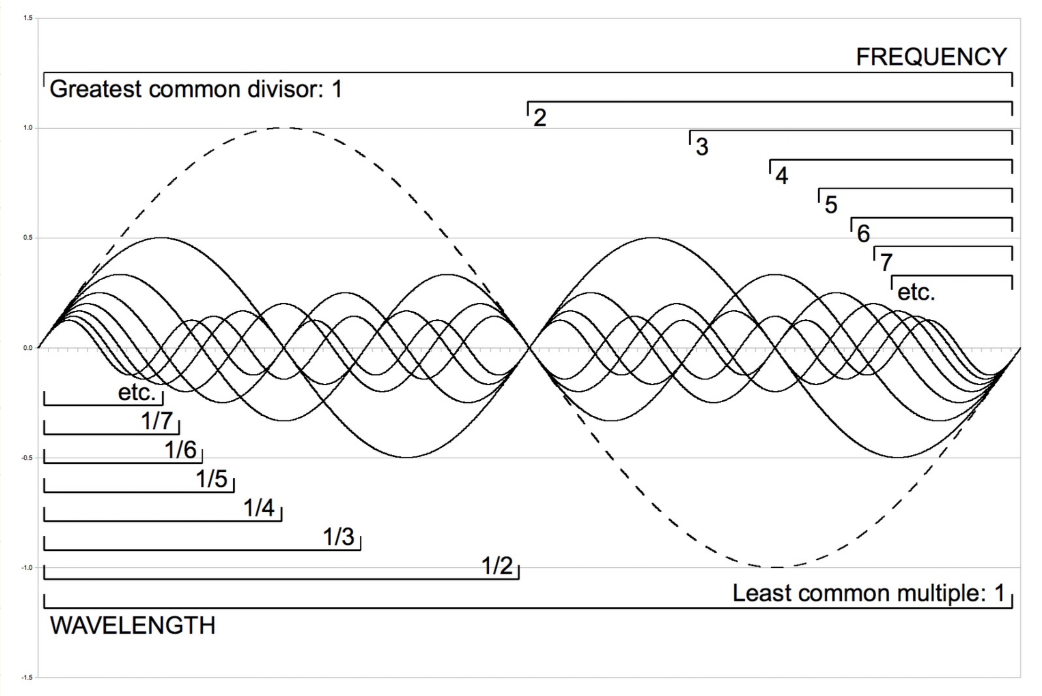
inverse wavelet transformation.

$$c_\psi = \left\{2\pi\int_{-\infty}^{\infty}\frac{|\hat{\psi}(\xi)|^2}{|\xi|}\,d\xi\right\}^{1/2} < \infty$$

the admissibility constant.

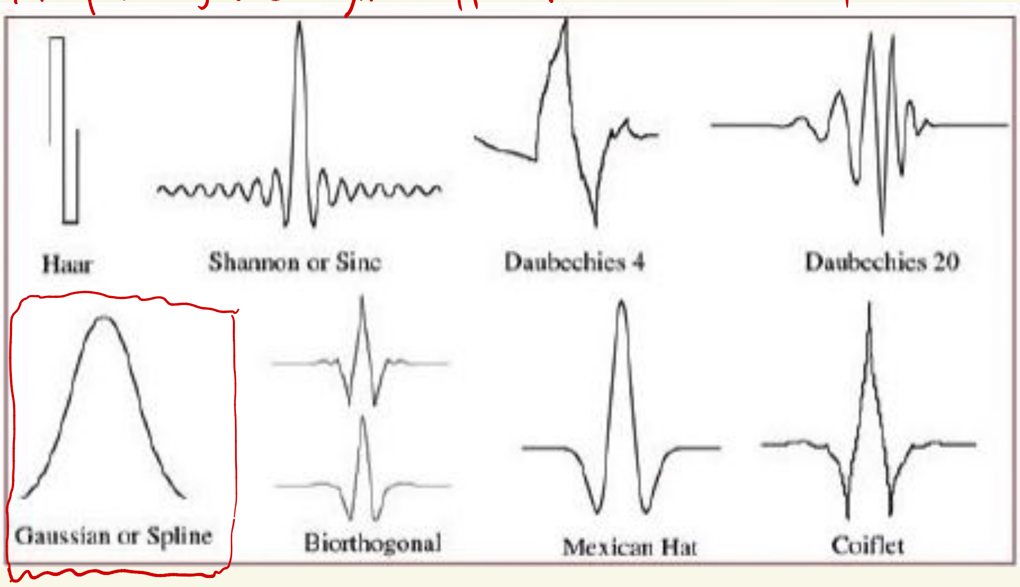
Fourier basis function vs. wavelet basis function.

The following is an example of fourier functions with different frequencies:



$\psi_f(t)$

The following are eight different mother wavelet functions.



$\psi_{s,\tau}(t)$

We will use Gaussion wavelet function in the following examples:

In the following example, we show step by step how to calculat $\Phi_x^\psi(\tau,s)$ for $\tau=2,40,90,140$ and $s=1,5,10$.

We use Gaussion Wavelet function:

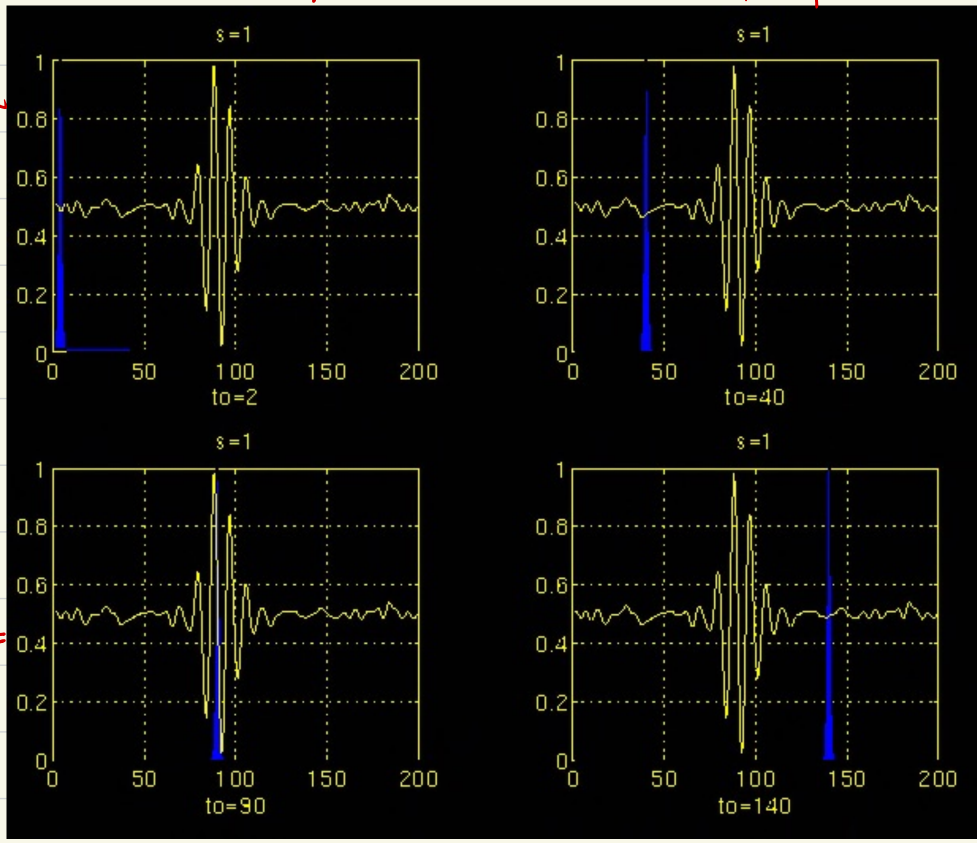
The wavelet at scale $s=$"1" is mupltiplied by the signal and then integrated over all times. The result of the integration is then multiplied by the constant number $1/\sqrt{s}$. The final result is the value of the transformation.

$\Phi_x^\psi(\tau=2,s=1) \Leftarrow$

$\Rightarrow \Phi_x^\psi(\tau=40,s=1)$

$\Phi_x^\psi(\tau=90,s=1) \Leftarrow$

$\Rightarrow \Phi_x^\psi(\tau=140,s=1)$

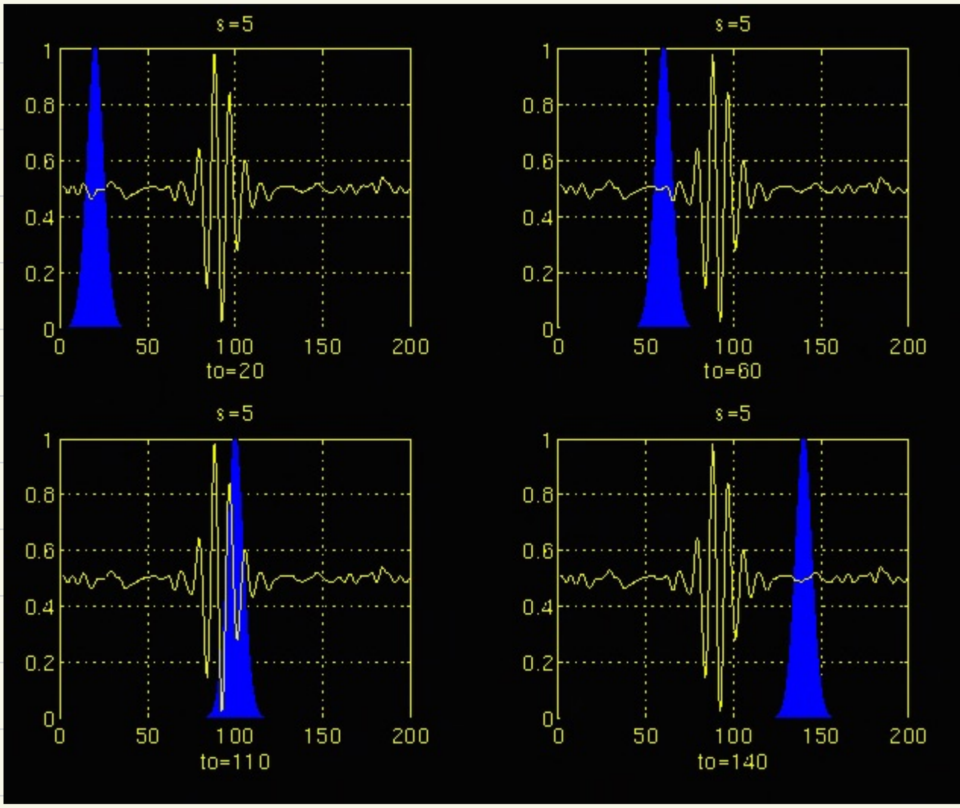


$\Phi_x^\psi(\tau=2,s=5) \Leftarrow$

$\Rightarrow \Phi_x^\psi(\tau=40,s=5)$

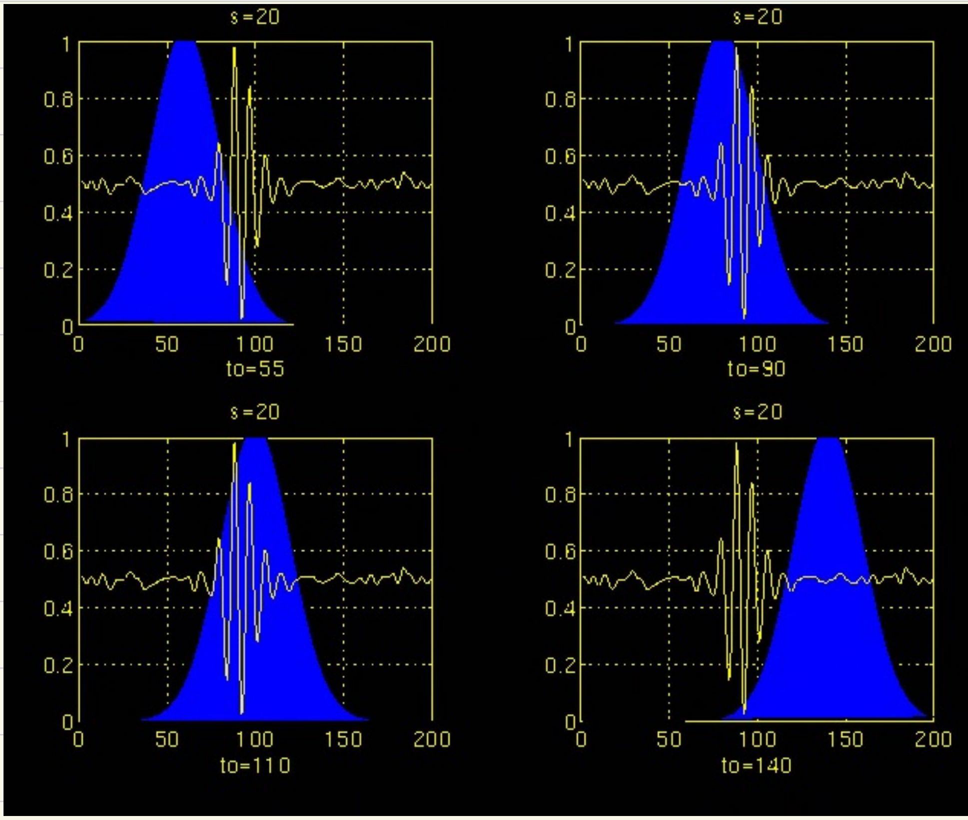
$\Phi_x^\psi(\tau=90,s=5) \Leftarrow$

$\Rightarrow \Phi_x^\psi(\tau=140,s=5)$



$\Phi_x^{\psi}(\tau=55, s=20) \Leftarrow$

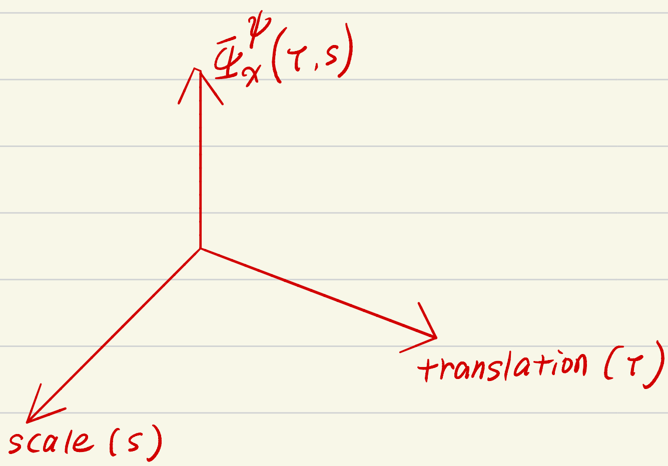
$\Rightarrow \Phi_x^{\psi}(\tau=90, s=20)$



$\Phi_x^{\psi}(\tau=110, s=20) \Leftarrow$

$\Rightarrow \Phi_x^{\psi}(\tau=140, s=20)$

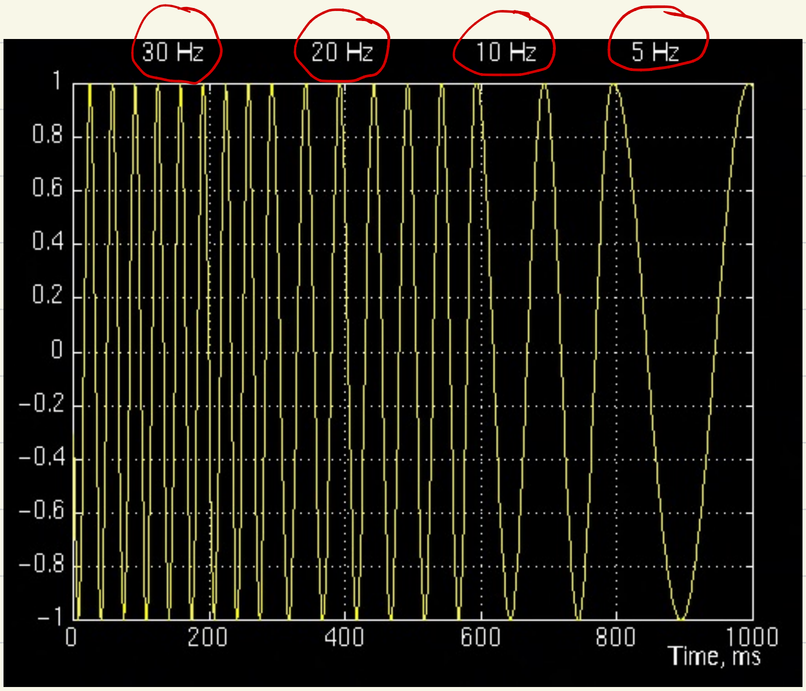
as shown in this example, the transformation value for each combination of (τ, s) can be calculated. we can draw the continuous wavelet transform (CWT) of this signal in a three dimension-al space: Scale(s) vs. Translation (τ) vs. amplitude $(\Phi_x^{\psi}(\tau, s))$.



Visualization of CWT of a signal

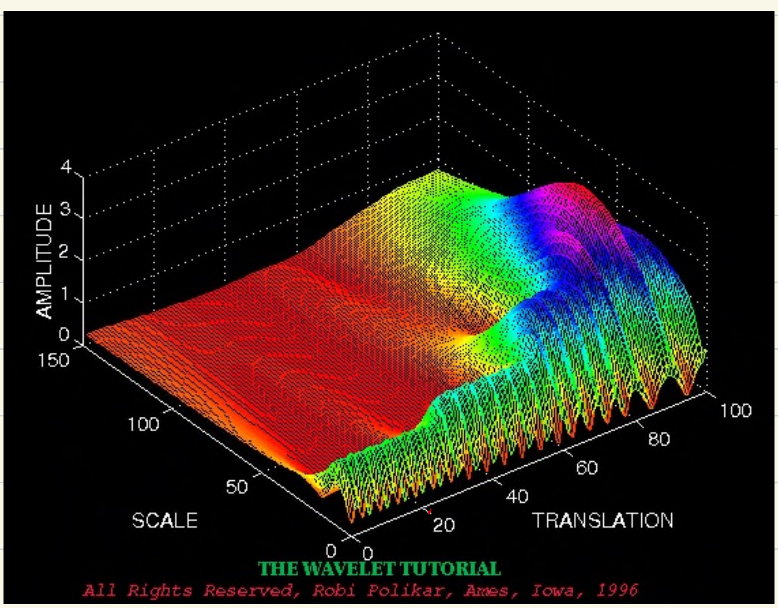
Consider the following non-stationary signal:

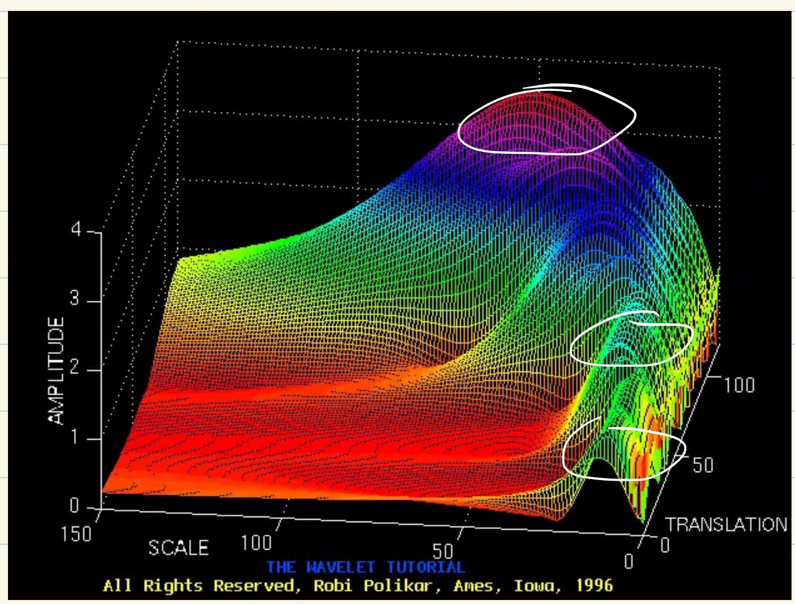
The signal is composed of four frequency Components at 30Hz, 20Hz, 10Hz, and 5 Hz.



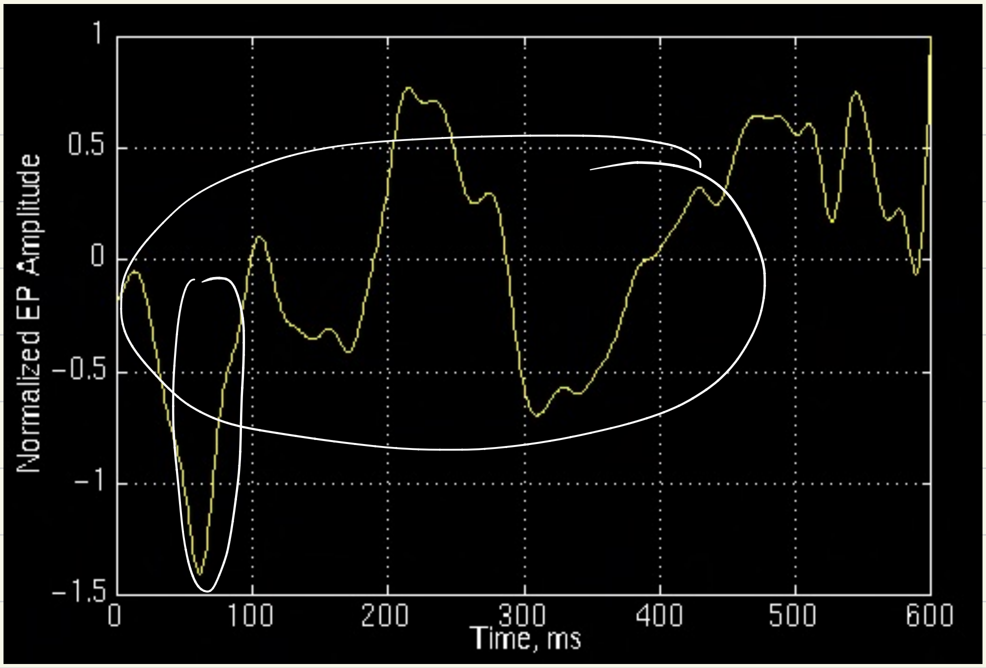
this signal has 30 Hz (highest frequency) components that appear at the lowest scale at a translation of 0 to 30. Then comes at the 20Hz Component, Second highest frequency, and so on. Then 5 Hz component appears at the end of the translation axis (as expected), and at higher scales (lowest frequencies) again as expected.

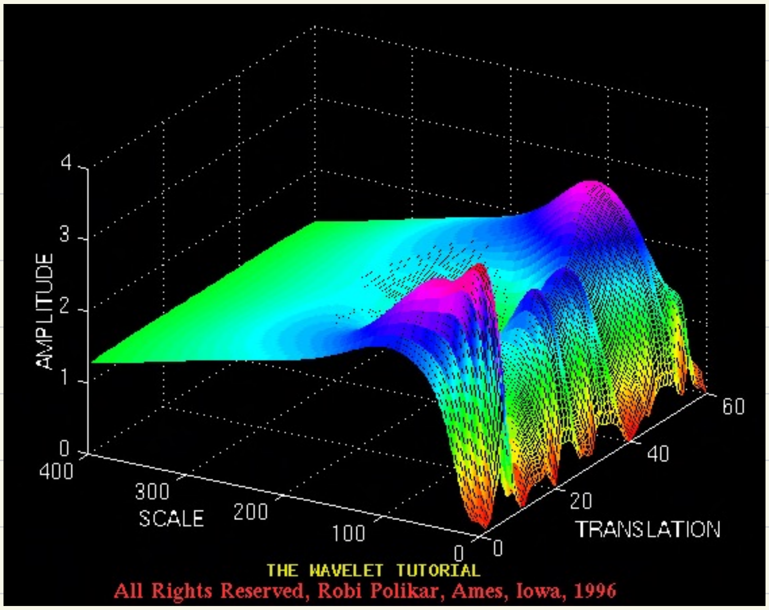
The following is the CWT of this signal:

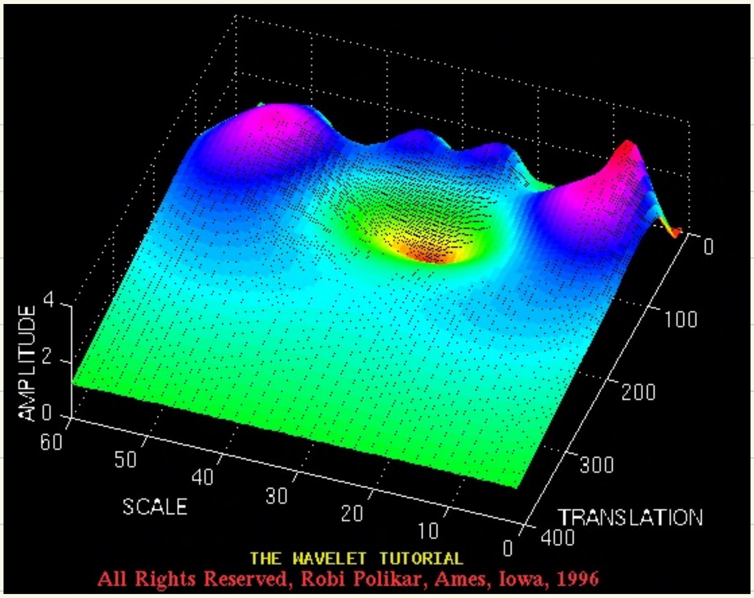




(different angle)

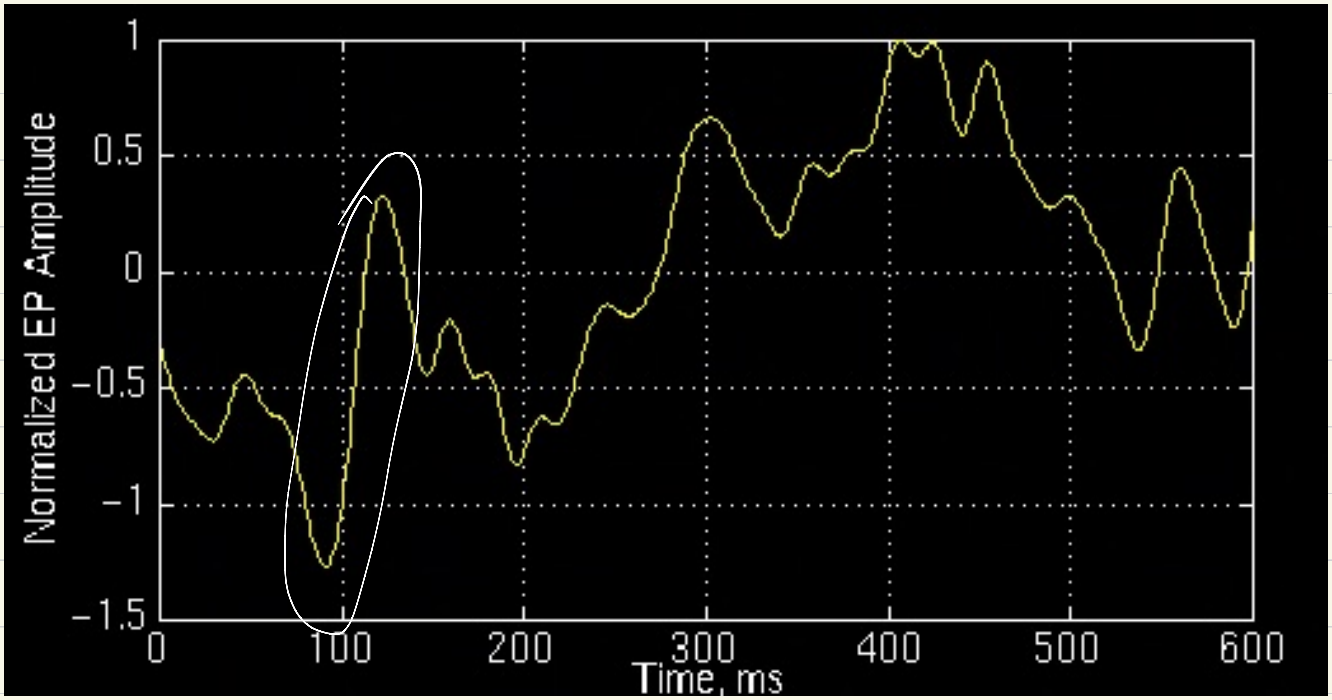


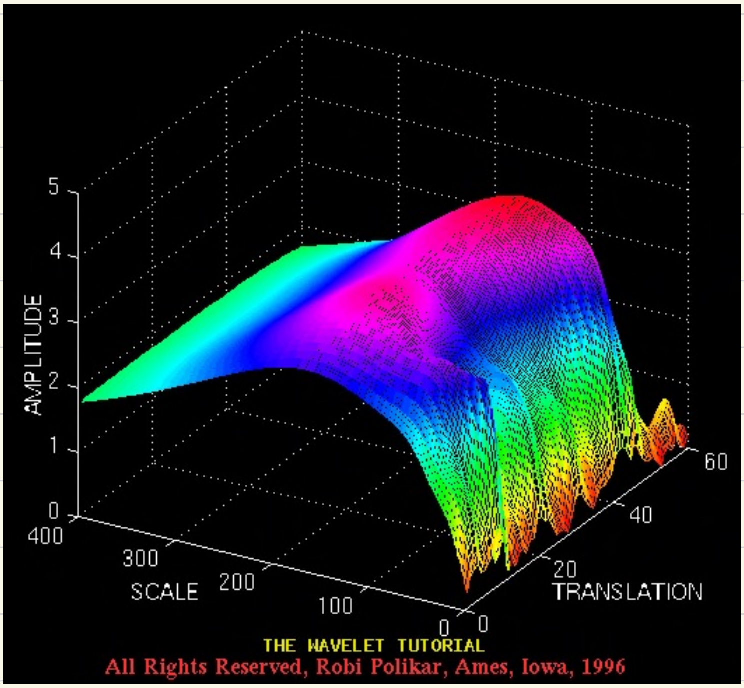


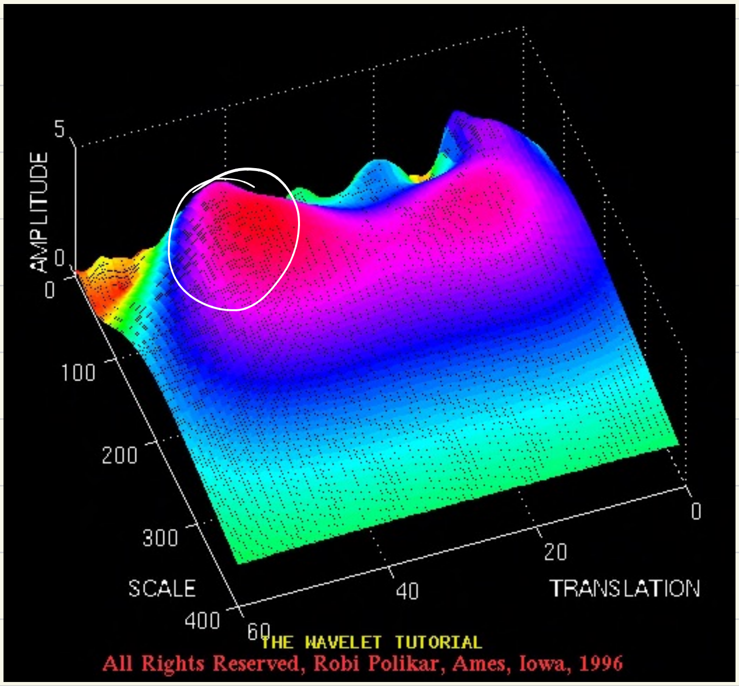


The left peak is located in a low-translation and high-scale region.

The right peak is located in a low-translation and low-scale region.







The peak has low-scale and low-translation.

Summary of continuous wavelet transform (CWT)

mother wavelet function

$$\boxed{\psi_{\tau,s}(t) = \frac{1}{\sqrt{|s|}}\,\psi\left(\frac{t-\tau}{s}\right)}$$

wavelet function

(wavelet transform)

$$\Phi_x^{\psi}(\tau,s) = \int x(t)\,\psi^{*}_{\tau,s}(t)\,dt$$

time domain (t) — spectrum domain (τ, s)

$x(t)$ — $\Phi_x^{\psi}(\tau,s)$

$$x(t) = \frac{1}{C_\psi^2}\int_s\int_\tau \Phi_x^{\psi}(\tau,s)\cdot\frac{1}{s}\,\psi_{\tau,s}(t)\,d\tau\,ds$$

(inverse wavelet transform)

$$\boxed{\psi_f(t) = e^{2j\pi f t}}$$

(Fourier transform)

$$X(f) = \int x(t)\cdot\psi_f^{*}(t)\,dt$$

time domain (t) — spectrum domain (f)

$x(t)$ — $\hat{x}(f)$

$$x(t) = \int X(f)\cdot\psi_f(t)\,df$$

(inverse fourier transform)