<u>Plasmas as Fluids</u>
At this point we need to use a number of basic equations that describe plasmas as fluids. While it is possible to calculate these equations from first principles, using Maxwell's electromagnetic field equations and Maxwell's velocity distribution (Man that guy did everything!) the process is tedious, fairly difficult and very time consuming. As the aim of this class is to develop a basic feel for plasmas (wisdom rather than book smarts) in this class we will simply assume that the equations are correct. For those of you who feel more adventuresome, notes going through those derivations are found at //www.utdallas.edu/~goeckner/plasma_sci_class/ Please realize that those derivations took ~ 4 class periods to complete.
The basic equations are follows:

<u>Boltzmann's Equation</u>
$$\frac{df}{dt} = 0 = (\nabla_r f)\Sigma \mathbf{v} + (\nabla_v f)\Sigma \mathbf{a} + \frac{\partial f}{\partial t}$$
This is relatively easy to prove…
$$\frac{df}{dt} = \frac{\partial f}{\partial \mathbf{r}}\frac{d\mathbf{r}}{dt} + \frac{\partial f}{\partial \mathbf{v}}\frac{d\mathbf{v}}{dt} + \frac{\partial f}{\partial t}$$
$$= (\nabla_r f)\Sigma \mathbf{v} + (\nabla_v f)\Sigma \mathbf{a} + \frac{\partial f}{\partial t}$$
$$= \mathbf{v}\,\Sigma(\nabla_r f) + \frac{\mathbf{F}}{m}\Sigma(\nabla_v f) + \frac{\partial f}{\partial t}$$

<u>Zeroth moment of the Boltzmann Equation – The Equation of Continuity (Particle conservation)</u>
$$\boxed{f\big|_c = \left(\nabla_r \,\Sigma(n\langle \mathbf{v}\rangle) + \frac{\partial n}{\partial t}\right)}$$
Moments are derived by multiplying by $\mathbf{v}^{moment}f(\mathbf{v})$ and integrating over all velocity. Thus what we are seeing is a measure of the 'average' of this particular parameter.

<u>First moment of the Boltzmann equation – Momentum Conservation</u>
This is also known as the <u>fluid equation of motion</u>
$$\boxed{mn\left(\frac{\partial \langle \mathbf{v}\rangle}{\partial t} + \langle \mathbf{v}\rangle \Sigma \nabla_r \langle \mathbf{v}\rangle\right) = \underbrace{\Delta \mathbf{M}\big|_c}_{\substack{momentum \\ lost\ via \\ collisions}} - \underbrace{m\langle \mathbf{v}\rangle f\big|_c}_{\substack{momentum\ change \\ via\ particle\ gain/loss}} - \nabla_r \Sigma \mathbf{P} + qn(\mathbf{E} + \langle \mathbf{v}\rangle \wedge \mathbf{B})}$$

<u>Poisson's Equation (which comes straight from Maxwell's E-M equations)</u>
$$\nabla^2 \Phi = -\frac{\rho}{\varepsilon}$$
$$= \frac{e}{\varepsilon}(n_e - n_i)$$

$$n = n_0 \exp\left(\frac{e\Phi}{kT_e}\right)$$

Now we have the basic equations (There are a few higher moments that come into play elsewhere that we will not have to deal with in this class.) we can begin to see how plasmas react as a whole. Remember plasmas have these 'collective' behaviors – which we are going to explore now. (We need to understand these behaviors so that we can understand many of the diagnostics used in plasmas.)

The first behavior that we will explore is the ability of the plasma to shield out static electric fields. This behavior should make sense as any strong electric field in a plasma will separate the negative and positive charge carriers. The distance over which a field can penetrate is known as the Debye length. (Note that electromagnetic fields can also be shielded out BUT the behavior is very different. We will get to this soon.)

Debye Length

We can now calculate the Debye length – an effective length over which a plasma will shield an electric field. (The length is the 1/e distance for reducing a potential.)
First, we have Poisson's equation

$$\nabla^2\Phi = -\frac{\rho}{\varepsilon}$$

$$= \frac{e}{\varepsilon}(n_e - n_i)$$

We make the further assumption that the density of the electrons in absence of the potential is the same as the ion density. (We make this assumption because if either the electrons or the ions were to leave an area, a significant electric field would be setup to try to pull them back together.) Thus,

$$n_i = n_0$$

Plugging this and the Boltzmann relation into Poisson's equation gives

$$\nabla^2\Phi = \frac{en_0}{\varepsilon}\left(\exp\left(\frac{e\Phi}{kT}\right) - 1\right)$$

Now, using the Taylor series expansion of the exponential, which we assume is approximately 1, gives

$$\nabla^2\Phi = \frac{en_0}{\varepsilon}\left(1 + \frac{e\Phi}{kT} + O\left(\left(\frac{e\Phi}{kT}\right)^2\right) - 1\right)$$

$$\approx \frac{e^2 n_0 \Phi}{\varepsilon kT}$$

Solving the differential equation leaves

$$\Phi \approx \Phi_0 \exp\left(\frac{-|\mathbf{r}|}{\lambda_{Debye}}\right) \quad where$$

$$\lambda_{Debye} = \left(\frac{e^2 n_0}{\varepsilon k T}\right)^{-\frac{1}{2}}$$

Bulk Motions

At this point, we need to deal with some of the bulk motions that occur in plasmas. These are not single particle motions but rather collective motion of all/most of the charge species in the plasma. The first, and most important is the electrostatic plasma oscillation, giving rise to the plasma frequency. [This but just one of a very wide variety of waves in plasmas.] These oscillations occur because one of the species becomes displaced from the other. When it accelerates back toward the other species, in gains too much energy and over shoots. To derive the plasma frequency, we will assume the simplest of geometries and plasmas.

1) No external fields. (This can be relaxed and the same result can be obtained.)
2) No random motion of the particles. (Hence, all particles of a species move at the same velocity at the same point in space. This can be relaxed and one can get the same result – it is just harder to do.)
3) Only the electrons move. (This is not a bad assumption for many aspects of plasmas.)
4) The plasma is one-dimensional and of such a length that the walls do not influence the result. (This implies that we are considering just regions that are at least several λ_{debye} from the walls.)

From Maxwell's equations we have,
$$\nabla \Sigma \mathbf{E} = \rho/\varepsilon$$
$$\nabla \wedge \mathbf{E} = 0$$
(We will ignore the induced magnetic field.) Then our equation of motion (momentum conservation) and continuity (particle conservation) become

Continuity Equation
$$f|_c = \left(\nabla_\mathbf{r} \Sigma(n\langle\mathbf{v}\rangle) + \frac{\partial n}{\partial t}\right)$$
$$\Downarrow$$
$$0 = \nabla_\mathbf{r} \Sigma(n\langle\mathbf{v}\rangle) + \frac{\partial n}{\partial t}$$

Energy Equation

$$mn\left(\frac{\partial \langle \mathbf{v}\rangle}{\partial t} + \langle \mathbf{v}\rangle \Sigma \nabla_\mathbf{r} \langle \mathbf{v}\rangle\right) = \underbrace{\Delta \mathbf{M}|_c}_{\substack{momentum\\ lost\ via\\ collisions}} - \underbrace{m\langle \mathbf{v}\rangle f|_c}_{\substack{momentum\ change\\ via\ particle\ gain/loss}} - \nabla_\mathbf{r} \Sigma \mathbf{P} + qn(\mathbf{E} + \langle \mathbf{v}\rangle \wedge \mathbf{B})$$

$$\Downarrow$$

$$mn\left(\frac{\partial \langle \mathbf{v}\rangle}{\partial t} + \langle \mathbf{v}\rangle \Sigma \nabla_\mathbf{r} \langle \mathbf{v}\rangle\right) = qn(\mathbf{E})$$

For this particular wave, we are considering deviations from charge neutrality. Thus we will have an induced electric field given by

$$\varepsilon \nabla \Sigma \mathbf{E} = \rho = e(n_i - n_e)$$

We have three items that are changing with time, $\mathbf{E}$, n_e and $<v_e>$. We will expand each of these items to produce a time average term, denoted with a '0' and an oscillating term denoted with a '1'. Thus

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1 \quad \text{- but } \mathbf{E}_0 \approx 0!$$

$$n_e = n_0 + n_1$$

$$\mathbf{v}_e = \mathbf{v}_0 + \mathbf{v}_1 \quad \text{- but } \mathbf{v}_0 \approx 0!$$

Then our conservation of momentum (energy) equation becomes

$$\left(\frac{\partial(\mathbf{v}_0 + \mathbf{v}_1)}{\partial t} + (\mathbf{v}_0 + \mathbf{v}_1)\Sigma\nabla_\mathbf{r}(\mathbf{v}_0 + \mathbf{v}_1)\right) = \frac{q}{m}(\mathbf{E}_0 + \mathbf{E}_1)$$

$$\left(\frac{\partial(\mathbf{v}_1)}{\partial t} + (\mathbf{v}_1)\Sigma\nabla_\mathbf{r}(\mathbf{v}_1)\right) = \frac{q}{m}(\mathbf{E}_1)$$

Now the second term on the left-hand side is small compared to the other two. (Two oscillating terms as opposed to one.) Thus we are left with

$$\frac{\partial(\mathbf{v}_1)}{\partial t} = \frac{q}{m}(\mathbf{E}_1) \quad \text{- let } \mathbf{v}_1 \propto e^{i(\omega t - \beta x)} \quad - \left(\begin{array}{c}\text{implying we have}\\ \text{a travelling wave}\end{array}\right)$$

$$\Downarrow$$

$$i\omega\mathbf{v}_1 = -\frac{e}{m}\mathbf{E}_1$$

Now we can do the same thing to the continuity equation

$$\nabla_\mathbf{r} \Sigma(n\langle \mathbf{v}\rangle) + \frac{\partial n}{\partial t} = 0$$

$$\nabla_\mathbf{r} \Sigma((n_0 + n_1)(\mathbf{v}_1)) + \frac{\partial(n_0 + n_1)}{\partial t} = 0$$

$$\nabla_\mathbf{r} \Sigma((n_0)(\mathbf{v}_1)) + \frac{\partial(n_1)}{\partial t} = 0$$

Where again we have dropped the higher order terms. Thus

$$n_0 \nabla_\mathbf{r} \Sigma(\mathbf{v}_1) = -\partial_t(n_1) \quad \text{- letting } n_1 \propto e^{i(\omega t - \beta x)}$$

$$\Downarrow$$

$$in_0 \beta \mathbf{v}_1 = -i\omega n_1$$

Finally, we solve Poisson's Equation
$$\varepsilon \nabla \Sigma \mathbf{E} = e(n_i - n_e)$$

$$\varepsilon \nabla \Sigma \mathbf{E}_1 = e(n_0 - (n_0 + n_1))$$

$$-i\beta\varepsilon E_1 = -en_1$$

This gives us three equations and three unknowns
$$in_0 \beta \mathbf{v}_1 = -i\omega n_1, \ i\beta\varepsilon E_1 = en_1, \ i\omega \mathbf{v}_1 = -\frac{e}{m}\mathbf{E}_1.$$

Combining the last two to eliminate E_1 gives
$$\mathbf{v}_1 = -\frac{e^2 n_1}{\omega m \beta \varepsilon}.$$

Placing this into the first to eliminate n_1 gives
$$\omega_{pe}^2 = \frac{n_0 e^2}{m\varepsilon}; \ \ \omega_{pe} = f_{pe}/2\pi,$$
the angular electron frequency of the plasma. This is also known as the dispersion relation.

Typically for process plasmas the density is $\sim 10^{10\text{-}12}$ cm^{-3}. Thus, $f_{pe} \sim 1$ to 10 GHz.

This is, in some sense, the simplest oscillation that can exist in a plasma. Note that this is not a wave in the typical sense! (Energy does not move in this oscillation, the group velocity, $\frac{d\omega}{d\beta}$, is zero. Here 'β' is the wave vector.)

There are numerous other oscillations that are waves that can transfer energy. They can be divided into electrostatic waves and electromagnetic waves. We will deal first with the electrostatic waves.

Electrostatic waves in plasmas
Let us go back to our fundamental fluid equations, the continuity equation, the energy equation, and Poisson's equation. As before we will ignore the collision terms and assume that the magnetic field is zero. Here however, we will include the pressure variations of the species.
Poisson's equation
$$\varepsilon \nabla \Sigma \mathbf{E} = \rho = e(n_i - n_e)$$
Continuity Equation

$$f|_c = \left(\nabla_{\mathbf{r}} \cdot \Sigma(n\langle\mathbf{v}\rangle) + \frac{\partial n}{\partial t} \right)$$

$$\Downarrow$$

$$0 = \nabla_{\mathbf{r}} \cdot \Sigma(n\langle\mathbf{v}\rangle) + \frac{\partial n}{\partial t}$$

Energy Equation

$$mn\left(\frac{\partial \langle\mathbf{v}\rangle}{\partial t} + \langle\mathbf{v}\rangle \cdot \Sigma\nabla_{\mathbf{r}}\langle\mathbf{v}\rangle \right) = \underbrace{\Delta\mathbf{M}|_c}_{\substack{momentum \\ lost \ via \\ collisions}} - \underbrace{m\langle\mathbf{v}\rangle f|_c}_{\substack{momentum \ change \\ via \ particle \ gain/loss}} - \nabla_{\mathbf{r}} \cdot \Sigma\mathbf{P} + qn(\mathbf{E} + \langle\mathbf{v}\rangle \wedge \mathbf{B})$$

$$\Downarrow$$

$$mn\left(\frac{\partial \langle\mathbf{v}\rangle}{\partial t} + \langle\mathbf{v}\rangle \cdot \Sigma\nabla_{\mathbf{r}}\langle\mathbf{v}\rangle \right) = qn(\mathbf{E}) - \nabla_{\mathbf{r}} \cdot \Sigma\mathbf{P}$$

This would be identical to the plasma oscillations except for the pressure term. We will deal with that term first.

We know from the ideal gas law that $p = nkT$. Then, assuming an isotropic, or one dimensional, plasma

$$\nabla_{\mathbf{r}} \cdot \Sigma\mathbf{P} = \nabla_{\mathbf{r}} p$$
$$= kT\nabla_{\mathbf{r}} n$$

This is true provided that the compression is 'isothermal'. In other words the temperature stays the same during the compression. Often, this is not true. Rather, we have 'adiabatic' compression, where the temperature changes. In this case, it can be shown using thermodynamics that

$$p = Cn^\gamma .$$

Here C is a constant and $\gamma = C_p/C_V$ is the ratio of the specific heats. We can see from the above equation that

$$\nabla p = \nabla\left(Cn^\gamma\right)$$
$$= C\nabla\left(n^\gamma\right)$$
$$= C\gamma n^{\gamma-1}\nabla n$$
$$= \gamma Cn^\gamma \frac{\nabla n}{n}$$
$$= \gamma p \frac{\nabla n}{n}$$

Further, it can be shown that $\gamma = \frac{N+2}{N}$ where N is the number of degrees of freedom.

Thus for N=1, $\gamma = 3$. (This is a crude approximation but it works for our needs.)

Thus, our equation of motion becomes

$$mn\left(\frac{\partial \langle\mathbf{v}\rangle}{\partial t} + \langle\mathbf{v}\rangle \cdot \Sigma\nabla_{\mathbf{r}}\langle\mathbf{v}\rangle \right) = qn(\mathbf{E}) - \gamma kT\nabla_{\mathbf{r}} n .$$

As before, we will assume that the density, velocity and electric field consists of a time-averaged term and an oscillating term. Thus,

$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1$ - but $\mathbf{E}_0 \approx 0$!

$n_e = n_0 + n_1$

$\mathbf{v}_e = \mathbf{v}_0 + \mathbf{v}_1$ - but $\mathbf{v}_0 \approx 0$!

letting

$\mathbf{v}_1, \mathbf{E}_1, n_1 \propto e^{i(\omega t - \beta x)}$

We can now follow the derivation that we made before but this time we will add our additional term.

Assuming that we are examining electrons, our conservation of momentum (energy) equation becomes

$$m(n_0 + n_1)\left(\frac{\partial\left(\overset{=0}{\mathbf{v}_0} + \mathbf{v}_1\right)}{\partial t} + \left(\overset{=0}{\mathbf{v}_0} + \mathbf{v}_1\right)\Sigma\nabla_{\mathbf{r}}\left(\overset{=0}{\mathbf{v}_0} + \mathbf{v}_1\right)\right) = -e(n_0 + n_1)\left(\overset{=0}{\mathbf{E}_0} + \mathbf{E}_1\right) - \gamma k T_e \nabla(n_0 + n_1)$$

$$mn_0 \frac{\partial(\mathbf{v}_1)}{\partial t} = -en_0\mathbf{E}_1 - \gamma k T_e \nabla n_1$$

$$\Downarrow$$

$$imn_0\omega\mathbf{v}_1 = -en_0\mathbf{E}_1 + i\gamma k T_e \beta n_1$$

Now we can do the same thing to the continuity equation

$$\nabla_{\mathbf{r}} \Sigma(n\mathbf{v}) + \frac{\partial n}{\partial t} = 0$$

$$\nabla_{\mathbf{r}} \Sigma\left((n_0 + n_1)(\mathbf{v}_1)\right) + \frac{\partial(n_0 + n_1)}{\partial t} = 0$$

$$\nabla_{\mathbf{r}} \Sigma\left((n_0)(\mathbf{v}_1)\right) + \frac{\partial(n_1)}{\partial t} = 0$$

$$n_0\nabla_{\mathbf{r}} \Sigma(\mathbf{v}_1) = -\partial_t(n_1)$$

$$\Downarrow$$

$$in_0\beta\mathbf{v}_1 = i\omega n_1$$

Finally, we solve Poisson's Equation

$\varepsilon\nabla \Sigma\mathbf{E} = e(n_i - n_e)$

$\varepsilon\nabla \Sigma\mathbf{E}_1 = e(n_0 - (n_0 + n_1))$

$-i\beta\varepsilon E_1 = -en_1$

This gives us three equations and three unknowns

$in_0\beta\mathbf{v}_1 = i\omega n_1$, $-i\beta\varepsilon E_1 = -en_1$, $imn_0\omega\mathbf{v}_1 = -en_0\mathbf{E}_1 + i\gamma k T_e \beta n_1$.

Combining the last two to eliminate E_1 gives

$$\mathbf{v}_1 = \left(\frac{e^2}{m\beta\varepsilon\omega} + \frac{\gamma k T_e \beta}{m n_0 \omega} \right) n_1$$

which is very similar to what we got before except we now have a second term.
Placing this into the first to eliminate n_1 gives

$$\omega^2 = \frac{e^2 n_0}{m\varepsilon} + \frac{\gamma k T_e}{m}\beta^2$$

$$= \omega_{pe}^2 + c_\gamma^2 \beta^2 \quad - \text{ where } c_\gamma^2 = \frac{\gamma k T_e}{m} \text{ is the electron sound speed}$$

Note that here the group velocity is non-zero, meaning that energy <u>can</u> be carried by the wave.

Now, let us assume that we are dealing with (positive) ions. Here, however, the electric field is determined by the electrons, not the ions. Thus, we need to replace the electric field with the gradient of the potential and use Boltzmann's relation on the electron density. Thus, our equations become

$$\nabla_{\mathbf{r}} \, \Sigma (n\mathbf{v}) + \frac{\partial n}{\partial t} = 0$$

$$\Downarrow$$

$$n_0 \beta \mathbf{v}_1 = \omega n_1$$

(as before)

$$m_i n \left(\frac{\partial \langle \mathbf{v} \rangle}{\partial t} + \langle \mathbf{v} \rangle \Sigma \nabla_{\mathbf{r}} \langle \mathbf{v} \rangle \right) = q n (\mathbf{E}) - \gamma k T \nabla_{\mathbf{r}} n$$

$$= -en(\nabla \phi) - \gamma k T_i \nabla_{\mathbf{r}} n$$

$$m_i (n_0 + n_1) \left(\frac{\partial \left(\overset{=0}{\overbrace{\mathbf{v}_0}} + \mathbf{v}_1 \right)}{\partial t} + \left(\overset{=0}{\overbrace{\mathbf{v}_0}} + \mathbf{v}_1 \right) \Sigma \nabla_{\mathbf{r}} \left(\overset{=0}{\overbrace{\mathbf{v}_0}} + \mathbf{v}_1 \right) \right) = -e(n_0 + n_1) \nabla \left(\overset{=0}{\overbrace{\phi_0}} + \phi_1 \right) - \gamma k T_i \nabla (n_0 + n_1)$$

$$m_i n_0 \frac{\partial (\mathbf{v}_1)}{\partial t} = -en_0 \nabla \phi_1 - \gamma k T_i \nabla n_1$$

$$\Downarrow$$

$$m_i n_0 \omega \mathbf{v}_1 = en_0 \beta \phi_1 + \gamma k T_i \beta n_1$$

Now, we can't use Poisson's equation but rather we assume that the change in local density can be modeled with Boltzmann's equation. E.g. that the local ion density is the same as the local electron density and that the electron density is given by

$n_i \approx n_e = n_0 e^{e\phi/kT_e}$ (This approximation causes some small error)

$$= n_0 e^{e\left(\overset{=0}{\cancel{\phi_0}}+\phi_1\right)/kT_e} = n_0 e^{e\phi_1/kT_e}$$

$$\approx n_0\left(1+\frac{e\phi_1}{kT_e}\right)$$

$$\Downarrow$$

$$n_1 = n_0\frac{e\phi_1}{kT_e}$$

This gives us our three equations and three unknowns

$$n_1 = n_0\frac{e\phi_1}{kT_e}, \quad n_0\beta\mathbf{v}_1 = \omega n_1, \quad m_i n_0\omega\mathbf{v}_1 = en_0\beta\phi_1 + \gamma kT_i\beta n_1$$

Putting together the first two gives

$$\mathbf{v}_1 = \frac{\omega}{\beta}\frac{e\phi_1}{kT_e}.$$

Plugging this and the first into the third gives

$$\frac{\omega^2}{\beta^2} = \frac{kT_e + \gamma kT_i}{m_i}$$

This is the ion-acoustic or ion-sound waves.

All of the above are just a few examples of electrostatic waves. There are many more electrostatic waves.

We will now deal briefly with electromagnetic waves. **It is important to note that we will only look at the very simplest cases. There are a wide variety of electromagnetic waves that are sustained in plasmas.**

First, standard light waves exist. This comes directly from Maxwell's equation.

$$\nabla^2\begin{pmatrix}\mathbf{E}\\\mathbf{H}\end{pmatrix} = \sigma\mu\partial_t\begin{pmatrix}\mathbf{E}\\\mathbf{H}\end{pmatrix} + \varepsilon\mu\partial_t^2\begin{pmatrix}\mathbf{E}\\\mathbf{H}\end{pmatrix}$$

$$|\mathbf{E}| = \pm\frac{|\mathbf{H}|}{\eta}; \quad \eta = \frac{i\mu\omega}{\gamma} = \sqrt{\frac{i\mu\omega}{\sigma+i\varepsilon\omega}} \quad \left(= \sqrt{\frac{\mu}{\varepsilon}} \ \ if \ \ \sigma = 0\right)$$

  $sign \ \det er\min ed \ by \ growth/decay$

  $(growth => -, decay => +)$

$$\beta = \frac{2\pi}{\lambda} = \frac{c}{\omega}; c = \frac{1}{\sqrt{\varepsilon\mu}}$$

In plasmas, this is not quite correct. What happens if the waves are interacting with the plasma? From Maxwell's equations we have

$$\nabla \wedge \mathbf{E} = -\partial_t \mathbf{B}$$

$$\nabla \wedge \mathbf{H} = \mathbf{J}_{free} + \partial_t \mathbf{D}$$

$$\nabla \Sigma \mathbf{D} = \rho_{free} = e(n_i - n_e) \approx 0$$

$$\nabla \Sigma \mathbf{B} = 0 ds$$

However the current is not zero. This changes the fields. To solve the problem, we will consider only the time varying components. Taking the curl of the first equation and the time derivative of the second equation gives

$$\nabla \wedge (\nabla \wedge \mathbf{E}) = \nabla(\nabla \Sigma \mathbf{E}) - \nabla^2 \mathbf{E} = -\partial_t \nabla \wedge \mathbf{B}$$

$$\mu^{-1}\partial_t \nabla \wedge \mathbf{B} = \partial_t \mathbf{J}_{free} + \partial_t^2 \varepsilon \mathbf{E}$$

We can combine these to give

$$\nabla(\nabla \Sigma \mathbf{E}) - \nabla^2 \mathbf{E} = -\mu\partial_t \mathbf{J}_{free} - \varepsilon\mu\partial_t^2 \mathbf{E}$$

$$\beta\left(\overbrace{\beta \Sigma \mathbf{E}}^{\propto \rho = 0} \right) + \beta^2 \mathbf{E} = -i\omega\mu\mathbf{J}_{free} + \varepsilon\mu\omega^2 \mathbf{E} \quad - \quad c^{-2} = \varepsilon\mu$$

$$\Downarrow$$

$$(\beta^2 - c^{-2}\omega^2)\mathbf{E} = -i\omega\mu\mathbf{J}_{free}$$

If the light is at a high frequency, then the ions are effectively fixed. Thus the current is almost entirely due to the motion of the electrons. Then the current can be given as

$$\mathbf{J}_{free} = -en_0 \mathbf{v}_{e1}$$

from the equation of motion

$$\mathbf{F} = m\frac{d\mathbf{v}_{e1}}{dt} = -e\mathbf{E}$$

so that

$$\mathbf{J}_{free} = -\frac{in_0 e^2 \mathbf{E}}{\omega m}$$

plugging this in to Maxwell's equation gives

$$(\beta^2 - c^{-2}\omega^2)\mathbf{E} = -\frac{n_0 \mu e^2 \mathbf{E}}{m}$$

$$\Downarrow$$

$$0 = \left(\omega^2 - c^2 \frac{n_0 \mu e^2}{m} - c^2\beta^2 \right)\mathbf{E}$$

$$= \left(\omega^2 - \frac{n_0 e^2}{\varepsilon m} - c^2\beta^2 \right)\mathbf{E}$$

$$= (\omega^2 - \omega_{pe}^2 - c^2\beta^2)\mathbf{E}$$

$$\Downarrow$$

$$\omega^2 = \omega_{pe}^2 + c^2\beta^2$$

This is the dispersion relation of electromagnetic waves in a plasma.

There is a very useful application of this dispersion relation.   This is using the change in wavelength to measure the plasma density

There is also a very interesting phenomenon that occurs, known as cutoff.