

# Lecture #8: Complete Partial Orders

CS 6371: Advanced Programming Languages

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The denotational semantics of loops is part of a more general mathematical theory of complete partial orders and continuous functions. Some of the basics of that theory are presented below, culminating in the Knaster-Tarski Fixed-Point Theorem. We use the Fixed-Point Theorem to prove that our denotational definition of `while` loops is a well-formed mathematical definition and constitutes the least fixed point of the functional  $\Gamma$ . We begin with important definitions.

**Definition:** A *partial order* (p.o.) is a set  $P$  on which there is a binary relation  $\sqsubseteq$  which is

- (i) reflexive:  $\forall p \in P . p \sqsubseteq p$ ,
- (ii) transitive:  $\forall p, q, r \in P . (p \sqsubseteq q) \wedge (q \sqsubseteq r) \implies (p \sqsubseteq r)$ , and
- (iii) antisymmetric:  $\forall p, q \in P . (p \sqsubseteq q) \wedge (q \sqsubseteq p) \implies (p = q)$ .

**Definition:** A p.o.  $(P, \sqsubseteq)$  has a *bottom* element  $\perp_P$  iff there exists an element  $\perp_P \in P$  such that for all  $p \in P$ ,  $\perp_P \sqsubseteq p$ .

Observe that  $(\Sigma \rightarrow \Sigma, \subseteq)$  is a partial order because the subset relation  $\subseteq$  is reflexive, transitive, and antisymmetric. The empty set  $\{\}$  (i.e., the partial function that is undefined for all inputs) is a bottom element of this partial order because the empty set is a subset of every set.

**Definition:** We say  $p \in P$  is an *upper bound* of a subset  $X \subseteq P$  iff  $\forall q \in X . q \sqsubseteq p$ .

Note that not every set of partial functions has an upper bound. For example, if  $f(\sigma) \neq g(\sigma)$ , then the set  $\{f, g\}$  has no upper bound because there is no function  $h$  such that  $f \subseteq h$  and  $g \subseteq h$ . However, for any two partial functions such that  $f \subseteq g$ ,  $g$  is an upper bound of  $\{f, g\}$ .

**Definition:** We say  $p$  is a *least upper bound* of  $X$ , written  $p = \bigsqcup X$ , if  $p$  is an upper bound of  $X$  and  $p \sqsubseteq q$  for all upper bounds  $q$  of  $X$ . We also denote the least upper bound of two elements  $p, q \in P$  as  $p \sqcup q$ .

In the above example,  $g$  is also a least upper bound for  $\{f, g\}$  because  $g = f \sqcup g$ .

**Definition:** An  $\omega$ -*chain* of a partial order  $(P, \sqsubseteq)$  is an infinite sequence  $p_0, p_1, \dots \in P$  such that  $p_0 \sqsubseteq p_1 \sqsubseteq \dots$ .

Recall that we proved in class that  $\perp \subseteq \Gamma(\perp) \subseteq \Gamma^2(\perp) \subseteq \dots$  is a family of nested subsets. Therefore,  $\perp, \Gamma(\perp), \Gamma^2(\perp), \dots$  is an  $\omega$ -chain for  $(\Sigma \rightarrow \Sigma, \subseteq)$ .

**Definition:** A partial order  $(P, \sqsubseteq)$  is a *complete partial order* (cpo) iff every  $\omega$ -chain  $p_0, p_1, \dots \in P$  has a least upper bound  $\bigsqcup_{i \geq 0} p_i \in P$ .

Observe that  $(\Sigma \rightarrow \Sigma, \sqsubseteq)$  is a cpo because for every  $\omega$ -chain, the infinite union of all partial functions in the chain is also a partial function in  $\Sigma \rightarrow \Sigma$ . That infinite union is a least upper bound of the chain. For example,  $\bigcup_{i \geq 0} \Gamma^i(\perp)$  is a least upper bound for the chain  $\perp, \Gamma(\perp), \Gamma^2(\perp), \dots \in \Sigma \rightarrow \Sigma$ .

**Definition:** A function  $f : P \rightarrow P$  is *monotonic* iff for all  $p, q \in P$ ,  $p \sqsubseteq q \implies f(p) \sqsubseteq f(q)$ .

*Theorem.* Functional  $\Gamma$  is monotonic.

The proof is simple, and is left as an exercise to the reader.

**Definition:** A function  $f : P \rightarrow P$  is *continuous* iff it is monotonic and for all  $\omega$ -chains  $p_0, p_1, \dots \in P$ , we have

$$\bigsqcup_{i \geq 0} f(p_i) = f\left(\bigsqcup_{i \geq 0} p_i\right)$$

*Theorem.* Functional  $\Gamma$  is continuous.

*Proof.* Let  $p_0, p_1, p_2, \dots \in \Sigma \rightarrow \Sigma$  be an arbitrary  $\omega$ -chain in cpo  $(\Sigma \rightarrow \Sigma, \sqsubseteq)$ . The proof that  $\Gamma$  is continuous consists of two parts: First we prove that if  $(\sigma, \sigma') \in \bigcup_{i \geq 0} \Gamma(p_i)$  then  $(\sigma, \sigma') \in \Gamma(\bigcup_{i \geq 0} p_i)$ . This proves that  $\bigcup_{i \geq 0} \Gamma(p_i) \subseteq \Gamma(\bigcup_{i \geq 0} p_i)$ . Next we prove that if  $(\sigma, \sigma') \in \Gamma(\bigcup_{i \geq 0} p_i)$  then  $(\sigma, \sigma') \in \bigcup_{i \geq 0} \Gamma(p_i)$ . This proves that  $\bigcup_{i \geq 0} \Gamma(p_i) \supseteq \Gamma(\bigcup_{i \geq 0} p_i)$ . We conclude therefore that  $\bigcup_{i \geq 0} \Gamma(p_i) = \Gamma(\bigcup_{i \geq 0} p_i)$ .

**Proof of  $\subseteq$  direction:** Let  $(\sigma, \sigma') \in \bigcup_{i \geq 0} \Gamma(p_i)$  be given. Thus, there exists  $n \geq 0$  such that  $(\sigma, \sigma') \in \Gamma(p_n)$ . Since  $p_n \subseteq \bigcup_{i \geq 0} p_i$ , it follows from the monotonicity of  $\Gamma$  that  $\Gamma(p_n) \subseteq \Gamma(\bigcup_{i \geq 0} p_i)$ . Therefore  $(\sigma, \sigma') \in \Gamma(\bigcup_{i \geq 0} p_i)$ .

**Proof of  $\supseteq$  direction:** Now instead let  $(\sigma, \sigma') \in \Gamma(\bigcup_{i \geq 0} p_i)$  be given. From the definition of  $\Gamma$ , we know there are two possible cases:

**Case 1:** If  $\mathcal{B}[b]\sigma = F$  then  $\sigma' = \sigma$ . Since  $\{(\sigma, \sigma) \mid \mathcal{B}[b]\sigma = F\}$  is a subset of  $\Gamma(x)$  for every set  $x$ , it follows that  $(\sigma, \sigma') \in \bigcup_{i \geq 0} \Gamma(p_i)$ .

**Case 2:** If  $\mathcal{B}[b]\sigma = T$  then  $\sigma' = (\bigcup_{i \geq 0} p_i)(\mathcal{C}[c]\sigma)$ . Thus,  $(\mathcal{C}[c]\sigma, \sigma') \in \bigcup_{i \geq 0} p_i$ , so there exists  $n \geq 0$  such that  $(\mathcal{C}[c]\sigma, \sigma') \in p_n$ . Since  $\mathcal{B}[b]\sigma = T$ , it follows from the definition of  $\Gamma$  that  $(\sigma, \sigma') \in \Gamma(p_n)$ . We conclude that  $(\sigma, \sigma') \in \bigcup_{i \geq 0} \Gamma(p_i)$ .  $\square$

**Definition:** Let  $f : P \rightarrow P$  be a continuous function on a cpo  $P$ . A *fixed point* of  $f$  is an element  $p \in P$  such that  $f(p) = p$ .

**Theorem** (Knaster-Tarski Fixed-Point Theorem): Let  $f : P \rightarrow P$  be a continuous function on a cpo  $P$  with bottom  $\perp$ . Then  $\bigsqcup_{i \geq 0} f^i(\perp)$  is a least fixed point of  $f$ .

From the fixed-point theorem we conclude that  $\bigcup_{i \geq 0} \Gamma^i(\perp)$  is a least fixed point of  $\Gamma$ .