Lecture #10: Fixed-point Induction Examples

CS 6371: Advanced Programming Languages

February 13, 2020

Exercise 1. *Consider the following recursively defined function $f : \mathbb{Z} \to \mathbb{Z}$.*

$$f(x) = (x{=}0 \to 0 \mid x{>}0 \to 2 - f(1-x) \mid x{<}0 \to f(-x))$$

Find a closed-form definition of f and prove your answer.

To find a closed-form definition (i.e., one that is non-recursive and does not use *fix*), it is often useful to define functional F and then construct the *graph* of the least fixed point of F. Recall that functional F is defined by

$$F(g) = \lambda x\,.\,(x{=}0 \to 0 \mid x{>}0 \to 2 - g(1-x) \mid x{<}0 \to g(-x))$$

The graph of the least fixed point of F is the set of input-output pairs that comprises $\mathit{fix}(F)$. We can construct it incrementally by applying F to itself starting with $\bot$:

$$\begin{aligned}
F^0(\bot) &= \{\}\\
F^1(\bot) &= \{(0,0)\}\\
F^2(\bot) &= \{(0,0),(1,2)\}\\
F^3(\bot) &= \{(-1,2),(0,0),(1,2)\}\\
F^4(\bot) &= \{(-1,2),(0,0),(1,2),(2,0)\}\\
F^5(\bot) &= \{(-2,0),(-1,2),(0,0),(1,2),(2,0)\}\\
F^6(\bot) &= \{(-2,0),(-1,2),(0,0),(1,2),(2,0),(3,2)\}\\
F^7(\bot) &= \{(-3,2),(-2,0),(-1,2),(0,0),(1,2),(2,0),(3,2)\}
\end{aligned}$$

As you can see, eventually a pattern starts to emerge. Function f appears to return 2 on odd inputs and 0 on even inputs. Thus, we conjecture that $f = h$ where h is the following closed-form definition:

$$h(x) = \begin{cases} 2 & \text{if } x \text{ is odd} \\ 0 & \text{if } x \text{ is even} \end{cases}$$

This does not constitute a proof; it is merely a conjecture. We can prove the $f \subseteq h$ half of the conjecture using fixed point induction.

Proof. Define property P by $P(g) \equiv \forall x \in g^{\leftarrow}\,.\, g(x){=}h(x)$. We wish to prove $P(f)$. Define functional F as above, and observe that $\mathit{fix}(F) = f$ by the definition of recursion. Thus, to prove $P(f)$ it suffices to prove $P(\mathit{fix}(F))$ by fixed-point induction.

Base Case: $P(\perp)$ holds vacuously.

Inductive Hypothesis: Assume that $P(g)$ holds for some arbitrary function g. That is, assume that $\forall x{\in}g^{\leftarrow} \,.\, g(x){=}h(x)$.

Inductive Case: We will prove that $P(F(g))$ holds. Let $x \in F(g)^{\leftarrow}$ be given. Looking at the definition of F, there are three cases to consider:

Case 1: Suppose $x = 0$. Then by definition of F, $F(g)(x) = 0 = h(x)$.

Case 2: Suppose $x > 0$. Then by definition of F, $F(g)(x) = 2 - g(1 - x)$. By inductive hypothesis, $g(1 - x) = 2$ if $1 - x$ is odd and 0 if $1 - x$ is even. If x is odd then $1 - x$ is even, so $g(1 - x) = 0$; thus $2 - g(1 - x) = 2 = h(x)$. If x is even then $1 - x$ is odd, so $g(1 - x) = 2$; thus $2 - g(1 - x) = 0 = h(x)$. Either way, $F(g)(x) = 2 - g(1 - x) = h(x)$.

Case 3: Suppose $x < 0$. Then by definition of F, $F(g)(x) = g(-x)$. By inductive hypothesis, $g(-x) = 2$ if $-x$ is odd and 0 if $-x$ is even. Since $-x$ has the same parity as x, it follows that $F(g)(x) = 2$ if x is odd and 0 if x is even. Hence, $F(g)(x) = h(x)$. □

Functions of multiple arguments can be treated as functions of a single pair argument.

Exercise 2. *Consider the following recursively defined function* $f : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$.

$$f(x, y) = (x{=}0 \to y \mid y{=}0 \to x \mid x, y{>}0 \to f(x - 1, y - 1) + 1)$$

Prove that $f \subseteq \max$.

Proof. Define property P by $P(g) \equiv \forall(x, y) \in g^{\leftarrow} \,.\, g(x, y){=}\max(x, y)$. We wish to prove $P(f)$. Define functional F in the usual way:

$$F(g) = \lambda(x, y)\,.\,(x{=}0 \to y \mid y{=}0 \to x \mid x, y{>}0 \to g(x - 1, y - 1) + 1)$$

To prove $P(f)$ it suffices to prove $P(\mathit{fix}(F))$ by fixed-point induction.

Base Case: $P(\perp)$ holds vacuously.

Inductive Hypothesis: Assume that $P(g)$ holds for some arbitrary function g. We will prove that $P(F(g))$ holds. Let $(x, y) \in F(g)^{\leftarrow}$ be given.

Case 1: Suppose $x = 0$. Then by definition of F, $F(g)(x, y) = y = \max(x, y)$.

Case 2: Suppose $y = 0$. Then by definition of F, $F(g)(x, y) = x = \max(x, y)$.

Case 3: Suppose $x, y > 0$. Then by definition of F, $F(g)(x, y) = g(x - 1, y - 1) + 1$. By inductive hypothesis, $F(g)(x) = \max(x - 1, y - 1) + 1$. If $x \geq y$ then $\max(x - 1, y - 1) = x - 1$, so $F(g)(x, y) = x - 1 + 1 = x$. If $x < y$ then $\max(x - 1, y - 1) = y - 1$, so $F(g)(x, y) = y - 1 + 1 = y$. In either case $F(g)(x, y) = \max(x, y)$. □