Complete Partial Orders

CS 6371: Advanced Programming Languages

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The denotational semantics of loops is part of a more general mathematical theory of complete partial orders and continuous functions. Some of the basics of that theory are presented below, culminating in the Knaster-Tarski Fixed-Point Theorem. We use the Fixed-Point Theorem to prove that our denotational definition of `while` loops is a well-formed mathematical definition and constitutes the least fixed point of the functional Γ. We begin with important definitions.

Definition: A *partial order* (p.o.) is a set P on which there is a binary relation $\sqsubseteq$ which is

(i) reflexive: $\forall p \in P \,.\, p \sqsubseteq p$,

(ii) transitive: $\forall p, q, r \in P \,.\, (p \sqsubseteq q) \wedge (q \sqsubseteq r) \Longrightarrow (p \sqsubseteq r)$, and

(iii) antisymmetric: $\forall p, q \in P \,.\, (p \sqsubseteq q) \wedge (q \sqsubseteq p) \Longrightarrow (p = q)$.

Definition: A p.o. $(P, \sqsubseteq)$ has a *bottom* element $\bot_P$ iff there exists an element $\bot_P \in P$ such that for all $p \in P$, $\bot_P \sqsubseteq p$.

Observe that $(\Sigma \rightharpoonup \Sigma, \subseteq)$ is a partial order because the subset relation $\subseteq$ is reflexive, transitive, and antisymmetric. The empty set $\{\}$ (i.e., the partial function that is undefined for all inputs) is a bottom element of this partial order because the empty set is a subset of every set.

Definition: We say $p \in P$ is an *upper bound* of a subset $X \subseteq P$ iff $\forall q \in X \,.\, q \sqsubseteq p$.

Note that not every set of partial functions has an upper bound. For example, if $f(\sigma) \neq g(\sigma)$, then the set $\{f, g\}$ has no upper bound because there is no function h such that $f \subseteq h$ and $g \subseteq h$. However, for any two partial functions such that $f \subseteq g$, g is an upper bound of $\{f, g\}$.

Definition: We say p is a *least upper bound* of X, written $p = \bigsqcup X$, if p is an upper bound of X and $p \sqsubseteq q$ for all upper bounds q of X. We also denote the least upper bound of two elements $p, q \in P$ as $p \sqcup q$.

In the above example, g is also a least upper bound for $\{f, g\}$ because $g = f \cup g$.

Definition: An ω-*chain* of a partial order $(P, \sqsubseteq)$ is an infinite sequence $p_0, p_1, \ldots \in P$ such that $p_0 \sqsubseteq p_1 \sqsubseteq \cdots$.

Recall that we proved in class that $\bot \subseteq \Gamma(\bot) \subseteq \Gamma^2(\bot) \subseteq \cdots$ is a family of nested subsets. Therefore, $\bot, \Gamma(\bot), \Gamma^2(\bot), \ldots$ is an ω-chain for $(\Sigma \rightharpoonup \Sigma, \subseteq)$.

Definition: A partial order $(P, \sqsubseteq)$ is a *complete partial order* (cpo) iff every ω-chain $p_0, p_1, \ldots \in P$ has a least upper bound $\bigsqcup_{i\geq 0} p_i \in P$.

Observe that $(\Sigma \rightharpoonup \Sigma, \subseteq)$ is a cpo because for every ω-chain, the infinite union of all partial functions in the chain is also a partial function in $\Sigma \rightharpoonup \Sigma$. That infinite union is a least upper bound of the chain. For example, $\bigcup_{i\geq 0} \Gamma^i(\bot)$ is a least upper bound for the chain $\bot, \Gamma(\bot), \Gamma^2(\bot), \ldots \in \Sigma \rightharpoonup \Sigma$.

Definition: A function $f : P \to P$ is *monotonic* iff for all $p, q \in P$, $p \sqsubseteq q \Longrightarrow f(p) \sqsubseteq f(q)$.

Theorem. Functional Γ is monotonic.

The proof is simple, and is left as an exercise to the reader.

Definition: A function $f : P \to P$ is *continuous* iff it is monotonic and for all ω-chains $p_0, p_1, \ldots \in P$, we have

$$\bigsqcup_{i\geq 0} f(p_i) = f\big(\bigsqcup_{i\geq 0} p_i\big)$$

Theorem. Functional Γ is continuous.

Proof. Let $p_0, p_1, p_2, \ldots \in \Sigma \rightharpoonup \Sigma$ be an arbitrary ω-chain in cpo $(\Sigma \rightharpoonup \Sigma, \subseteq)$. The proof that Γ is continuous consists of two parts: First we prove that if $(\sigma, \sigma') \in \bigcup_{i\geq 0} \Gamma(p_i)$ then $(\sigma, \sigma') \in \Gamma\big(\bigcup_{i\geq 0} p_i\big)$. This proves that $\bigcup_{i\geq 0} \Gamma(p_i) \subseteq \Gamma\big(\bigcup_{i\geq 0} p_i\big)$. Next we prove that if $(\sigma, \sigma') \in \Gamma\big(\bigcup_{i\geq 0} p_i\big)$ then $(\sigma, \sigma') \in \bigcup_{i\geq 0} \Gamma(p_i)$. This proves that $\bigcup_{i\geq 0} \Gamma(p_i) \supseteq \Gamma\big(\bigcup_{i\geq 0} p_i\big)$. We conclude therefore that $\bigcup_{i\geq 0} \Gamma(p_i) = \Gamma\big(\bigcup_{i\geq 0} p_i\big)$.

Proof of $\subseteq$ direction: Let $(\sigma, \sigma') \in \bigcup_{i\geq 0} \Gamma(p_i)$ be given. Thus, there exists $n \geq 0$ such that $(\sigma, \sigma') \in \Gamma(p_n)$. Since $p_n \subseteq \bigcup_{i\geq 0} p_i$, it follows from the monotonicity of Γ that $\Gamma(p_n) \subseteq \Gamma\big(\bigcup_{i\geq 0} p_i\big)$. Therefore $(\sigma, \sigma') \in \Gamma\big(\bigcup_{i\geq 0} p_i\big)$.

Proof of $\supseteq$ direction: Now instead let $(\sigma, \sigma') \in \Gamma\big(\bigcup_{i\geq 0} p_i\big)$ be given. From the definition of Γ, we know there are two possible cases:

Case 1: If $\mathcal{B}[\![b]\!]\sigma = F$ then $\sigma' = \sigma$. Since $\{(\sigma, \sigma) \mid \mathcal{B}[\![b]\!]\sigma = F\}$ is a subset of $\Gamma(x)$ for every set x, it follows that $(\sigma, \sigma') \in \bigcup_{i\geq 0} \Gamma(p_i)$.

Case 2: If $\mathcal{B}[\![b]\!]\sigma = T$ then $\sigma' = \big(\bigcup_{i\geq 0} p_i\big)(\mathcal{C}[\![c]\!]\sigma)$. Thus, $(\mathcal{C}[\![c]\!]\sigma, \sigma') \in \bigcup_{i\geq 0} p_i$, so there exists $n \geq 0$ such that $(\mathcal{C}[\![c]\!]\sigma, \sigma') \in p_n$. Since $\mathcal{B}[\![b]\!]\sigma = T$, it follows from the definition of Γ that $(\sigma, \sigma') \in \Gamma(p_n)$. We conclude that $(\sigma, \sigma') \in \bigcup_{i\geq 0} \Gamma(p_i)$. $\square$

Definition: Let $f : P \to P$ be a continuous function on a cpo P. A *fixed point* of f is an element $p \in P$ such that $f(p) = p$.

Theorem (Knaster-Tarski Fixed-Point Theorem): Let $f : P \to P$ be a continuous function on a cpo P with bottom $\bot$. Then $\bigsqcup_{i\geq 0} f^i(\bot)$ is a least fixed point of f.

From the fixed-point theorem we conclude that $\bigcup_{i\geq 0} \Gamma^i(\bot)$ is a least fixed point of Γ.