A Closed System of Production Possibility and Social Welfare

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Abstract

We offer a closed system production possibility and social welfare system that can be modeled using virtually any available software package. It has the attribute that social welfare is not independent of production possibilities. The closure is made using the famous result by Negishi (1960) for a purely competitive economy. The goal is to help students to understand the interaction, through experimentation, between production and social choice in a competitive economy.

1 Introduction

Most microeconomic textbooks demonstrate how a social optimal output can be obtained by the use of production possibility curves (PP) and a social indifference curve. They frequently begin by deriving, diagrammatically, the PP by use of the Edgeworth-Bowley Box diagram. This is usually accomplished by drawing two sets of isoquants for two products that compete for the limited inputs, e.g., labor L and capital stock K. If the isoquants are drawn carefully so that there are always pairs of tangent isoquants for the two products, and if they are numbered to represent the output levels of both, then the PP curve can be drawn. Depending on the numbering of the isoquants, different shaped PP curves will result. The usual shape is one where the PP curve is strongly concave from below. In actuality, of course, it is more likely the case that PP curves are only slightly concave. This is because in a perfectly competitive economy the ex-post equilibrium is one where most firms are producing at approximately lowest long run average cost where constant returns to scale exist.

The next pedagogical step is to assume that there is some social welfare function that yields a set of social indifference curves. These social indifference curves tend to look like an individual's indifference curves but they are assumed to be some underlying composite of all individuals' preferences. The student is also reminded at this time that the social welfare function can't satisfy Arrow's Impossibility axioms. Nevertheless, every society must seek to optimize some type of social welfare function, subject to resource constraints, even if they don't know what the function looks like. The graphical solution of the entire system of PP curve and social indifference curves usually shows a PP curve that is strongly concave from below and is tangent to a social indifference curve that is strongly convex from below.

The instructor realizes that the simplicity of this stylized picture hides the many complications and interactions involved in obtaining an optimal point on the PP curve. From the student's perspective there is a logical gap between drawing a point of tangency between the social indifference and the PP curve, and understanding how it could actually be derived. They realize that there has to be some connection between earned income, individual preferences and the tangency point. There are some advanced economic texts that attempt to investigate these topics in much greater detail.[1] Yet the analyses in such texts are usually beyond the ability of upper level undergraduates or even many first year graduate students. The increased familiarity students now have with personal computers and with mathematical software like Excel and Mathcad, however, make investigations through simulation a viable option.[2]

Here we describe a set of equations that and be easily simulated and which provide a closed system for obtaining the interaction between the PP curve and an associated social curve that is generated from the PP curve. Since perfect competition is the standard benchmark to which other systems are compared, we make use of the famous result by Negishi (1960). Negishi

[1]These include texts by Varian (1992, Ch. 17.8, 22.2), Luenberger (1995, Ch. 6, 10), and Mas-Colell, Whinston, and Greene (1995, Ch. 17 and Appendix). The older intermediate text by Layard and Walters (1978) provided a useful integration of these topics.

[2]Afriat's (1987, Ch. III.4) book used BASIC programming to simulate solutions to the problems discussed here. Murphy (1995) used DERIVE to produce a PP curve. We use Mathcad and Excel since they are more familiar and widely available to students.

proved that a perfectly competitive economy acts as if it were seeking to maximize a particular social welfare function subject to the equilibrium supply and demand considerations and to the production capabilities of the economy. The social welfare function is the weighted sum of the individual utility functions, where the respective weights are the reciprocals of the marginal utilities of income of the individuals. Negishi's result has been reproduced and clearly presented in advanced texts (see footnote 2). It has also played a crucial role in the debate on the theory of the second best.[3] As we shall demonstrate in our simulations, the Negishi social indifference curves do not always conform to the nice stylized pictures given in economic texts. Nevertheless, they are the benchmark results of the perfectly competitive economy and the complete simulation of these, along with the PP curve, allows the student to investigate a wide array of possible outcomes.

2 The Production Possibility Curve

We begin with two Cobb-Douglas type production functions for products Q and Y.[4] These are both functions of labor L and capital stock K. The total amount of labor and capital stock are $\overline{L}$ and $\overline{K}$. If all labor and capital stock are used in the two productions, we have:

$$Q = AL^{\alpha}K^{\beta} \tag{1}$$

$$Y = B(\overline{L} - L)^{a}(\overline{K} - K)^{b} \tag{2}$$

Taking the total differential of both (1) and (2) and setting dQ and dY equal to zero, we obtain the slopes of the isoquants for both products. Setting the slopes equal to each other, as would be the case in the Edgeworth-Bowley box diagram, we arrive at the equation:

$$\frac{\alpha}{\beta} \cdot \frac{K^{1-\beta}}{L^{1-\alpha}} = \frac{a}{b} \cdot \frac{(\overline{K} - K)^{1-b}}{(\overline{L} - L)^{1-a}} \tag{3}$$

Equation (3) does not have an explicit solution for K as a function of L and the production function parameters. However, for any set of parameters, α, β, a, and b, and value of Labor L, the value of capital stock K, can be

[3] See Hamlen 2002 for the history and the relevance of Negishi's (1960) result in the Second Best literature, particularly in the debate between McManus (1959) and Davis and Whinston (1967).

[4] We have undertaken the analyses using the CES and translog production functions. It is more difficult, in a pedogogical sense, to choose meaningful parameters for these latter production functions.

obtained using a root equation solver. Thus some $K = g(L)$ relationship can be obtained. Equations (1) and (2) become:

$$Q(L) = AL^{\alpha}[g(L)]^{\beta} \tag{4}$$

$$Y(L) = B(\overline{L} - L)^{a}(\overline{K} - g(L))^{b} \tag{5}$$

As L ranges from zero to $\overline{L}$, the set of $Q = Q(L)$ and $Y = Y(L)$ can be calculated. These yield the PP curve. We will assume that $Q(L)$ is on the vertical axis and $Y(L)$ is on the horizontal axis. The slope of the PP curve can be approximated numerically, using equations (4) and (5) by taking the ratio:

$$v(L) = \frac{\Delta Q(L)}{\Delta Y(L)} = -\frac{P_Y}{P_Q} \tag{6}$$

The slope of the PP curve is important because it also represents the price ratio of the two goods $\frac{P_Y}{P_Q}$ at any point on the PP curve. In the current context we can set P_Q equal to one and allow variations in P_Y to capture the changes in the price ratio. In this case: $v(L) = -P_Y$.

3 The Social Welfare Function

We will assume that labor and capital stock owners are the two representative individuals in this economy. They each have Cobb-Douglas type utility curves that are functions of the two goods. These are given by:

$$U(Labor) = DQ^{s}Y^{t} \quad (\text{where } D, s, t > 0) \tag{7}$$

$$V(Capital\ Stock) = EQ^{w}Y^{x} \quad (\text{where } E, w, x > 0) \tag{8}$$

For simplicity, we allow $D = 1$ and $E = 1$.

Each individual maximizes utility subject to an income constraint, e.g., M_L (labor's income) and M_K (capital stock's income). Using equations (7) and (8) we can write the indirect utility functions as (Varian 1992, p. 102):

$$U(Labor) = M_L^{s+t}[P_Q^{-s}P_Y^{-t}(s+t)^{-s-t}s^{s}t^{t}] \tag{9}$$

$$U(Capital\ Stock) = M_K^{w+x}[P_Q^{-w}P_Y^{-x}(w+x)^{-w-x}w^{w}x^{x}] \tag{10}$$

As noted above, a competitive economy can be viewed as one that acts as if it maximizes the weighted sum of the individual utility functions subject to the standard production capabilities and supply and demand equations. The weights are the reciprocals of the marginal utility of income of the individual at the optimal solution. If we take the reciprocals of the marginal utilities using equations (9) and (10) and multiply them by their respective utilities, and again maintaining the above assumptions, we obtain the social welfare function that is the weighted sum of the two individuals' incomes:

$$W = \frac{M_L}{s+t} + \frac{M_K}{w+x} \tag{11}$$

If we set $s + t = 1$ and $w + x = 1$, maximizing social welfare requires maximizing the total level of income. In this sense we can understand why use of the gross domestic product as a measure of social welfare is at least reasonable. It hides, of course, the potential disparities between labor and capital stock owners.

We also know that in a competitive economy all inputs should receive the value of the marginal product of the last input hired. With Cobb-Douglas type production functions the shares are equal to powers of the respective inputs. For example, labor's income becomes:

$$M_L = \alpha P_Q Q + a P_Y Y = \alpha Q + a P_Y Y \qquad \text{(with } P_Q = 1\text{)} \tag{12}$$

A similar equation could be made for the owners of the capital stock. When there are constant returns to scale in both goods we know that the shares of income to both labor and capital stock owners would perfectly exhaust the value of total output. Also, in this case the PP curve would be a straight line. This is not an unexpected case since in perfectly competitive economies all firms eventually produce at their long run (ex-post) lowest average cost and this is approximately at the point in the production process where there are constant returns to scale. In the more likely case, the economy is not assumed to always be in a long run (ex-post) equilibrium, and most industries will be experiencing slightly increasing or decreasing returns to scale. In this case, however, payment based on the value of the marginal product of the last unit hired results in total payments not being equal to the total value of output. Therefore it is necessary to make some assumption on the destiny of the surplus or shortage. The most common assumption made, and the one used here, is that the capital stock owners are also the entrepreneurs and therefore receive the surplus or bear the loss. Thus our representative owner of capital stock receives the following income:

$$M_K = (1 - \alpha)Q + (1 - a)P_Y Y \tag{13}$$

Combining equations (4), (5), (11), (12), and (13) we have:

$$W(L) = Q(L)\left[\frac{1-\alpha}{s+t} + \frac{\alpha}{w+x}\right] - v(L)\,Y(L)\left[\frac{1-\beta}{s+t} + \frac{\beta}{w+x}\right] \tag{14}$$

If we add the common assumption that each individual's utility function is invariant up to a monotonic transformation, then we can assume that $(s + t) = 1$ and $(w + x) = 1$. In this case equation (14) becomes:

$$W(L) = Q(L) - v(L)\,Y(L) \tag{15}$$

In order to draw the social indifference curve using equation (15) we need to add the optimal supply equals demand conditions. These are obtained by solving the basic problem of maximizing utility subject to an income constraint. By using the income equations (12) and (13) along with the first order conditions for optimal selections of good by both labor and capital stock owners, we can derive a single supply equals demand condition:

$$\frac{\alpha\,Q(L) + a\,Y(L)\frac{P_Y}{P_Q}}{\frac{s}{t}+1} + \frac{(1-\alpha)\,Q(L) + (1-a)\,Y(L)\frac{P_Y}{P_Q}}{\frac{w}{x}+1} - Y(L) = 0 \tag{16}$$

In most cases there will be only one combination of $Q(L)$ and $Y(L)$ on the PP curve that solves equation (16). At this solution the optimal welfare function $W(L)^*$ can be calculated using equation (15). Then solving equation (15) for Q as a function of W^* and Y we have Negishi's optimal social indifference curve:

$$Q = W^* - \frac{P_Y}{P_Q}Y \tag{17}$$

In equation (17) the price ratio $-\frac{P_Y}{P_Q}$ is the slope of the PP curve at the optimal solution. For small variations around the optimal solution the price ratio may be regarded as fixed and the social indifference curve is a striaght-line tangent to the PP curve. If we consider larger variations around the optimal solution we must allow the price ratio to vary and the social indifference curve given by equation (17) can take on any shape, depending on the curvature of the PP curve. Therefore the social indifference curve for the Negishi solution does not always behave as the stylized social indifference curves shown in the textbooks. This is because it is not independent of the PP curve and the relative price ratio. For example, with strong decreasing returns to scale in both products the social indifference can intersect the PP curve at the optimal solution. Points that lie above the PP curve cannot be attained due to production constraints and points below the PP curve will not represent an equilibrium solution where supply equals demand. In the ex-post long run, however, we expect only mild increasing or decreasing returns to scale and the social indifference curve acts more like the traditional textbook case, as shown in the following simulation.

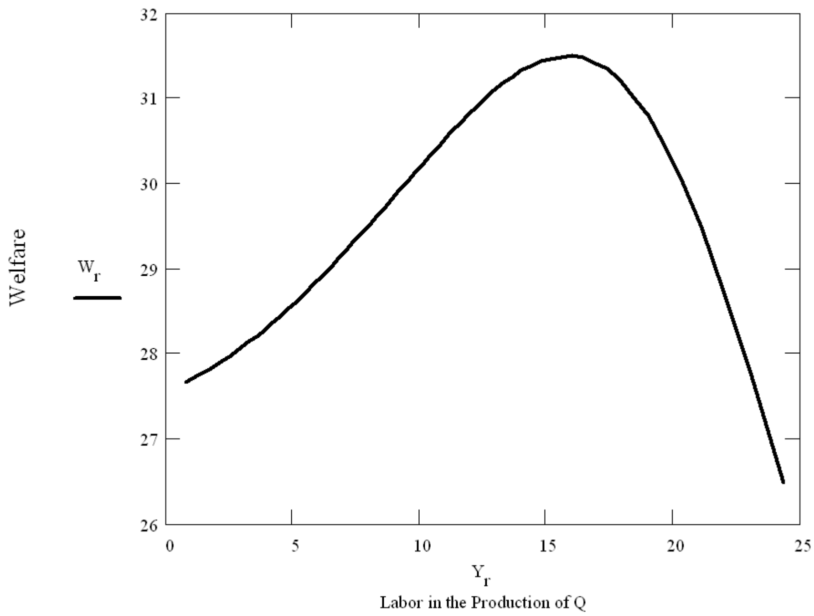

Figure 1: Social welfare function

4 The Simulation

We have simulated the above equations using both Mathcad and Excel.[5] Figures 1 and 2 provided here are from the Mathcad simulation of the welfare function and the PP curve and social indifference curve, respectively. The system, however, can be simulated using any of the available software packages that include root equation solver for equation (3). We use the following parameters: $\alpha = .45$, $\beta = .4$, $a = .6$, $b = .25$, $s = .2$, $t = .3$, $x = .5$, and $w = .5$. The total amount of labor and capital are: $\overline{L} = 60$ and $\overline{K} = 40$. Figure 1 shows the social welfare function, given by equation 15, as L goes from zero (where all labor is used to produce good Y) to $\overline{L}$ (where all labor is used to produce good Q). Figure 2 shows the resulting PP curve and social indifference curve for the optimal solution where $Q = 14.3$, $Y = 15.2$ and $\frac{P_Y}{P_Q} = 1.13$.

The parameters can be changed to reveal how different returns to scale or different consumer parameters affect the optimal solution. Also, it is not

[5] Our use of Mathcad solver as an instruction tool is identical to Hazera's (2005) conclusion, i.e., while we are enthusiastic about Mathcad's ease of use, the students are more familiar with Excel.

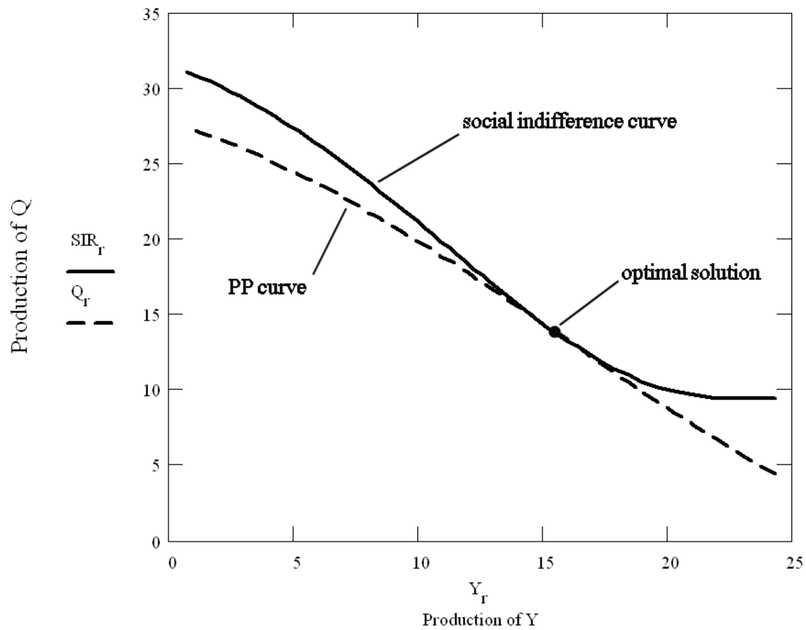

Figure 2: Social indifference curve and PP curve

difficult to determine how labor and capital stock owners are benefitted or hurt either in absolute or relative terms. This can be in terms of utility or income since they are closely related in the Negishi solution. Since we are using only two goods, two inputs, and two representative consumers, it is fairly easy to obtain boundary solutions where the optimal solution is to produce only one of the goods. We have also been able to construct two different countries, obtain the closed form solution for each, and then develop the international trade possibilities between the two. An elaboration of these equations is beyond the scope of the current article and has been accomplished in other articles where the social utility function is not related to production.

5 Conclusion

The primary value of this system of equations is that the interaction between the social indifference curve and the PP curve represents a benchmark solution—that of a perfectly competitive economy. It shows that the social welfare function is inherently related to the production and income generating equations underlying the perfectly competitive economy. By varying the parameters in the system, the instructor is also able to demonstrate how the competitive economy yields different relative remunerations to those who provide the labor and those who own the capital stock.

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