The Binomial Distribution
The Binomial and Sign Tests

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1 Overview

The binomial distribution models repeated choices between two alternatives. For example, it will give the probability of obtaining 5 Tails when tossing 10 coins or the probability for a rat to choose 10 times out of 20 the correct branch of a 3-branch maze. The binomial test uses the binomial distribution to decide if the outcome of an experiment using a binary variable (also called a *dichotomy*) can be attributed to a systematic effect. The sign test is applied to before/after designs and uses the binomial test to evaluate if the direction of change between before and after the treatment is systematic.

2 Binomial distribution

The binomial distribution models experiments in which a repeated binary outcome is counted. Each binary outcome is called a *Bernoulli* trial, or simply a trial. For example, if we toss 5 coins, each

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binary outcome corresponds to H or T, and the outcome of the experiment could count the number of T out of these 5 trials.

2.1 Notations and definitions

We call Y the random variable counting the number of outcomes of interest, N the total number of trials, P the probability of obtaining the outcome of interest on each trial and C a given number of outcomes. For example, if we toss 4 coins and count the number of Heads, Y counts the number of Heads, $N = 4$, and $P = \frac{1}{2}$. If we want to find the probability of getting 2 heads out of 4, then $C = 2$.

With these notations, the probability of obtaining C outcomes out of N trials is given by the formula

$$\Pr(Y = C) = \binom{N}{C} \times P^C \times (1-P)^{N-C} . \tag{1}$$

The term $\binom{N}{C}$ gives the number of combinations of C elements from an ensemble of N, it is called the “binomial of N, and C”, and is computed as

$$\binom{N}{C} = \frac{N!}{C!(N-C)!} \text{ where } N! = 1 \times 2 \cdots \times N . \tag{2}$$

For example, if the probability of obtaining 2 Heads when tossing 4 coins is computed as:

$$\begin{aligned}\Pr(Y = 2) &= \binom{N}{C} \times P^C \times (1-P)^{N-C} \\ &= \binom{4}{2} P^2 (1-P)^{4-2} \\ &= 6 \times .5^2 \times (1-.5)^2 = 6 \times .5^4 = .3750 .\end{aligned} \tag{3}$$

The mean and standard deviation of the binomial distribution are equal to

$$\mu_Y = N \times P \text{ and } \sigma_Y = \sqrt{N \times P \times (1-P)} . \tag{4}$$

The binomial distribution converges to the normal distribution for large values of N (practically, for $P = \frac{1}{2}$ and $N = 20$, the convergence is achieved).

3 Binomial test

The Binomial test uses the binomial distribution to decide if the outcome of an experiment in which we count the number of times one of two alternatives has occurred. For example, suppose we ask 10 children to attribute the name "keewee" or "koowoo" to a pair of dolls identical except for the size, and that we predict that children will choose keewee for the small doll. We found that 9 children out of 10 chose keewee. Can we consider that children choose sytematically? To answer this question, we need to evaluate the probability of obtaining 9 keewees or more than 9 keewees if the children were choosing randomly. If we denoted this probability by p, we find (from Equation 1) that:

$$\begin{aligned} p &= \Pr(9 \text{ out of } 10) + \Pr(10 \text{ out of } 10) \\ &= \binom{10}{9} \times P^{9} \times (1-P)^{10-9} + \binom{10}{10} \times P^{10} \times (1-P)^{0} \\ &= (10 \times .5^{9} \times .5^{1}) + (1 \times .5^{10} \times .5^{0}) \\ &= .009766 + .000977 \\ &\approx .01074\,. \end{aligned} \tag{5}$$

Assuming an alpha level of $\alpha = .05$, we can conclude that the children did not answer randomly.

3.1 $P \neq \frac{1}{2}$

The binomial test can be used with values of P different from $\frac{1}{2}$. For example, the probability p of having 5 rats choosing the correct door out of 4 possible doors in a maze, uses a values of $P = \frac{1}{4}$, and is equal to

$$\begin{aligned} p &= \Pr(6 \text{ out of } 6) + \Pr(5 \text{ out of } 6) \\ &= \binom{6}{6} \times P^6 \times (1-P)^{6-6} + \binom{6}{5} \times P^5 \times (1-P)^{6-5} \\ &= \frac{1}{4^6} + 6 \times \frac{1}{4^5} \times \frac{3}{4} = \frac{1}{4^6} + \frac{18}{4^6} \\ &\approx .0046\,. \end{aligned} \tag{6}$$

And we will conclude that the rats are showing a significant preference for the correct door.

3.2 Large N: Normal approximation

For large values of N, a normal approximation can be used for the Binomial distribution. In this case, p is obtained by first computing a Z score. For example, suppose that we had asked the doll question to 86 children and that 76 of them choose keewee. Using Equation 4, we can compute the associated Z-score as

$$Z_Y = \frac{Y - \mu_Y}{\sigma_Y} = \frac{76 - 43}{4.64} \approx 7.12\,. \tag{7}$$

The probability associated to such a value of Z being smaller than $\alpha = .001$, we can conclude that children did not answer randomly.

3.3 Sign test

The sign test is used in repeated measurement designs that measure a dependent variable on the same observations before and after some treatment. It tests if the direction of change is random or not. The change is expressed as a binary variable taking the values + is the dependent variable is larger for a given observation after the treatment and − if it is smaller. When there is not change, the change is coded 0 and is ignored in the analysis. For example, suppose that we measure the number of candies eaten on two different days by 15 children, and that we expose the children to a film showing the danger of eating too much sugar between these two days. On the second day, out of these 15 children, 5 ate the same number of candies, 9 ate less, and 1 ate more. Can we consider that the film diminished candies consumption? This problem is equivalent to comparing 9 positive outcomes against one negative with $P = \frac{1}{2}$. From Equation 5, we get that such a result has a p value smaller that $\alpha = .05$ and we conclude that the film did change the behavior of the children.

References

[1] Siegel, S. (1956) *Nonparametric statistics for the behavioral sciences.* New York: MacGraw-Hill.